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INVESTIGATION OF THE EFFECTS OF NONHOMOGENEOUS (OR NONSTATIONARY) BEHAVIOR ON THE SPECTRA OF ATMOSPHERIC TURBULENCE

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16. Abstract

This report examines the effects of nonhomogeneous or nonstationary envelope behavior on the power spectra of atmospheric turbulence records. The principal vehicle used in the study is a new series expansion of the instantaneous power spectrum that has for its first term the usual quasi-stationary spectrum approximation. The minimum duration of a burst of turbulence and the minimum rise-time of an abrupt onset of turbulence that will not give rise to changes in the spectrum due to the nonstationary behavior are determined. A general criterion for envelope behavior that will not give rise to changes in the spectrum is also determined. Spectra computed from recorded turbulence time-histories are shown to be consistent with the theoretical predictions.

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LIST OF SYMBOLS

a_n	expansion coefficient
B	bandwidth, beta function
b_m	expansion coefficient
C	constant
D	turbulence range interval
f	frequency, Hz
h	impulse response function
k	wavenumber
L	integral scale
ℓn	natural logarithm
M	half interval of Papoulis correlation window
m	mean value
P	probability distribution function
p	probability density function
P_m	Legendre polynomial
$P_m^{(j)}$	j*th* derivative of Legendre polynomial
R	autocorrelation function, remainder term
r	remainder terms
t	time
V	aircraft speed
w	vertical turbulence velocity
x	spatial coordinate
y	aircraft response, dummy variable
z	stationary Gaussian process

LIST OF SYMBOLS *(Cont.)*

Greek Symbols:

Γ	Gamma function
ν	dummy variable
ξ	dummy variable
σ	standard deviation
τ	time delay
ϕ	autocorrelation function
Φ	power spectrum

INTRODUCTION AND SUMMARY

For some time, the effects of nonhomogeneous behavior on the spectra of atmospheric turbulence records have been debated. See, for example, Fig. 19 and the accompanying discussion in the review paper "Atmospheric Turbulence" by Houbolt [*1*]. It can be argued that for homogeneous, isotropic turbulence, the von Karman spectrum should provide a good fit to experimentally determined spectra [*1*,*2*]. In some situations, an excellent fit has been obtained; in others, experimentally determined spectra have exhibited a considerably more rounded "knee" than would be predicted by the von Karman spectrum. This rounding of the knee has been attributed by some investigators to nonhomogeneous behavior of the turbulence records from which the spectra were computed.

The purpose of the work reported herein has been to determine under what conditions nonhomogeneous behavior can be expected to cause measurable deviations from the spectra that would have been obtained from "comparable" homogeneous records. This statement can be given a concrete meaning by assuming that the turbulence velocity records under consideration have the form*

$$w(x) = \sigma(x)\ z(x) \qquad (1.1a,\ 3.1a)$$

$$\langle z^2 \rangle = 1 \quad . \qquad (1.1b,\ 3.1b)$$

In the above equations, x denotes a spatial coordinate that may be obtained from records measured as a function of time t by using Taylor's hypothesis, i.e., $x = Vt$, where V is the speed of the aircraft making the measurements. The function $z(x)$ is assumed to be drawn from a homogeneous (i.e., stationary) random process with unit variance, and the nonhomogeneous standard deviation $\sigma(x)$ may be regarded as either a deterministic function or a sample function drawn from a random process. The function $\sigma(x)$ is, by definition, nonnegative. A random process $w(x)$ that has the form of Eq. (1.1) has been given the name "uniformly modulated" [*3*,*4*].

If the process $z(x)$ in Eq. (1.1) is assumed to be Gaussian in addition to being homogeneous, then its power spectral density can be measured by "infinitely clipping" (i.e., hard clipping) a

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*Most of the equations in this introductory section have two numbers. The first number designates the order of appearance of the equation in the present section; the second number designates the number associated with the same equation, as it appears later in the report where the material is treated in detail.

turbulence record $w(x)$ that is assumed to have the form of Eq. (1.1). This may be seen by first recognizing that, since $\sigma(x)$ is nonnegative, the zero crossings of $w(x)$ and $z(x)$, and the signs of $w(x)$ and $z(x)$, both must be identical. We can, therefore, construct the autocorrelation function of the sample function $z(x)$ from the infinitely clipped sample function $w(x)$ by using the so-called arcsin law [5].

$$\phi_z(\xi) = \sin\left[\frac{\pi}{2}\, \phi_0(\xi)\right] \quad , \tag{1.2, 3.20}$$

where $\phi_0(\xi)$ is the autocorrelation function of the infinitely clipped sample function $w(x)$ (with amplitudes of the hard clipped version of $w(x)$ set equal to plus or minus unity after clipping), and where $\phi_z(\xi)$ is the autocorrelation function of $z(x)$. The model described by Eq. (1.1) therefore provides us with a method of measuring the form of the spectrum of a homogeneous record $z(x)$ that is "comparable" to the nonhomogeneous record $w(x)$ except for the nonhomogeneous behavior; this spectrum is obtained by forming the Fourier transform of $\phi_z(\xi)$, which may be unambiguously computed from the record $w(x)$ by infinite clipping and using Eq. (1.2). If the shape of the spectrum obtained in this fashion is, except for statistical variations, the same as the shape of the spectrum computed from the nonhomogeneous record $w(x)$ in the usual way, then we must conclude that the nonhomogeneous behavior of the variance $\sigma^2(x)$ has had no measurable effect on the spectrum of $w(x)$. The spectra of three experimental records have been computed in both of the above ways in the work reported on herein.

Using assumptions or estimates of the function $\sigma(x)$ in Eq. (1.1), or of its autocorrelation function, we may compute the spectrum of $w(x)$ from the spectrum of $\sigma(x)$ and the spectrum of $z(x)$, where the spectrum of $z(x)$ is obtained by clipping $w(x)$ and using the "arcsin law" correction as explained above. Since we are dealing with nonhomogeneous processes, the method used in this report is to compute the instantaneous spectrum [6,7] of the process $w(x)$. The instantaneous spectrum possesses the important property that its time average yields the usual power spectral density of the process being described. Thus, potential effects of time-localized nonstationary behavior will show up clearly in the instantaneous spectrum; whereas, in computing the time average associated with the usual power spectrum, these localized effects can be averaged out. An additional advantage of the instantaneous spectrum description of nonstationary processes is that it possesses exact input-response relationships — e.g., for computation of the instantaneous spectra of aircraft responses [7]. Relevant properties of the instantaneous spectrum are reviewed in Sec. 2 of this report. General expressions for the instantaneous spectra of the class of nonhomogeneous processes $w(x)$ defined by Eq. (1.1) are dervied in Sec. 3 for cases where $\sigma(x)$

is regarded as a deterministic function, a homogeneous random process, and an ergodic process.

To ascertain the importance of the fluctuating behavior of $\sigma(x)$ on the instantaneous spectrum of $w(x)$, a new series expansion of the instantaneous spectrum of $w(x)$ is derived in Sec. 4. The first term in this series expansion is the usual quasi-homogeneous spectrum representation

$$\Phi_w(k,x) \approx \sigma^2(x)\ \Phi_z(k) \quad , \tag{1.3}$$

where $\Phi_z(k)$ is the power spectrum of the homogeneous component $z(x)$ of the process defined by Eq. (1.1). The coefficient of the *nth* term in the expansion is shown to be proportional to $(L_z/L_\sigma)^n$, where L_z is the integral scale of the homogeneous component $z(x)$ of the turbulence, L_σ is a length scale associated with the modulating function $\sigma(x)$, and where the quasi-homogeneous approximation provided by the first term is counted as $n = 0$. Thus, whenever $\sigma(x)$ varies slowly in comparison with $z(x)$ the expansion of the instantaneous spectrum $\Phi_w(k,x)$ will converge quickly.

Since the spectral form of the first term described by Eq. (1.3) is the same as that of the process $z(x)$, we can determine whether the nonhomogeneous behavior of the process $w(x)$ is sufficiently rapid to affect the spectrum by looking at the behavior of second term of the series expansion of $\Phi_w(k,x)$. The first two terms of the expansion may be expressed as

$$\Phi_w(k,x) \approx \sigma^2(x)\left[\Phi_z(k) - \frac{1}{16\pi^2}\ \frac{d^2\ell n\sigma(x)}{dx^2}\ \Phi_z^{(2)}(k)\right] \quad , \tag{1.4, 4.53}$$

where $\Phi_z^{(2)}(k)$ denotes the second derivative of the power spectrum $\Phi_z(k)$ of the process $z(x)$. Thus, if $|d^2[\ell n\sigma(x)]/dx^2|$ is sufficiently large, the shape of the spectrum will be affected by the nonhomogeneous behavior of $\sigma(x)$. In Sec. 4, it is shown that the integral with respect to x of the second term in the right-hand side of Eq. (1.4) always has the effect of smoothing the knee, whenever $\Phi_z(k)$ is the (transverse) von Karman spectrum. It is also shown there that the shape of the von Karman spectrum will, for practical purposes, be unchanged by the nonhomogeneous behavior whenever

$$L_z^2\left|\frac{d^2\ell n\sigma(x)}{dx^2}\right| \le 0.04 \quad , \tag{1.5, 4.82}$$

where L_z is the integral scale of the stationary component $z(x)$.

For the case where the homogeneous component z(x) in Eq. (1.1) is assumed to have a von Karman transverse spectrum, the two-term expansion of Eq. (1.4) is evaluated in Sec. 4 for two analytical forms of $\sigma(x)$, which represent an abrupt onset of turbulence and a burst of turbulence. In the example of the abrupt onset of turbulence, let L_σ denote the nominal "rise length" (i.e., V times the rise time) of the modulating function $\sigma(x)$. For this case, it is shown that when $L_\sigma \geq 10\ L_z$, the effects of the nonhomogeneous behavior will be virtually undetectable in the instantaneous spectrum; whereas, when $L_\sigma \leq 5\ L_z$, the instantaneous spectrum will display a strongly rounded knee for values of x in the vicinity of the rise of $\sigma(x)$.

In the example of the burst of turbulence, let L_σ denote approximately one-half of the total length of the burst. For this case, it is shown in Sec. 4 that when $L_\sigma \geq 13\ L_z$ the effects of the nonhomogeneous behavior of $\sigma(x)$ will be virtually undetectable in the instantaneous spectrum; whereas, when $L_\sigma \leq 7\ L_z$ the instantaneous spectrum will display a strongly rounded knee for all values of x.

The vertical trace of the velocity record displayed in Fig. 11 of this report has six "bursts" of turbulence identified by arrows on the record. According to the above criteria, it is shown in Sec. 4 that the first two and last two bursts shown on the record will cause strong smoothing of the knee of the von Karman transverse spectrum; whereas. the middle two bursts would probably cause weak but detectable smoothing of the knee. Bursts with durations appreciably longer than the durations of the middle two bursts shown in Fig. 11 would cause undetectable smoothing of the knee.

To verify the above theoretical conclusions, we processed two nonhomogeneous turbulence records using the techniques developed during the program. Our predictions indicated that the nonhomogeneous behavior of neither record occurred sufficiently rapidly to have a measurable effect on its spectrum. This prediction was experimentally verified, since the spectra computed from the infinitely clipped records, after correction using the "arcsin law" of Eq. (1.2) above, were essentially identical with the spectra computed in the usual way from the records w(x). The nonhomogeneous modulating function $\sigma(x)$ of one of the records was then length scaled, and the spectra computed from this scaled version of the original record were shown to be in complete agreement with the conclusions drawn from the burst-of-turbulence example described above.

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RELEVANT PROPERTIES OF THE INSTANTANEOUS POWER SPECTRUM

The method used in this report to study the effects of nonstationary behavior on the spectra of turbulence is based on the instantaneous power spectral density. The instantaneous spectrum and some of its properties are reviewed in the present section. The development follows closely the work of Mark [7].

The Instantaneous Power Spectrum

Consider a record w(t) of turbulence velocities. We shall assume that w(t) is a sample function from a generally nonstationary random process {w(t)}. Following Bendat and Piersol [6] and Mark [7], we may define an instantaneous autocorrelation function for the process {w(t)} as

$$\phi_w(\tau,t) \triangleq \langle w(t-\tfrac{\tau}{2})\ w(t+\tfrac{\tau}{2})\rangle \quad , \tag{2.1}$$

where the angular brackets shall denote everywhere in this report an ensemble average.* Again following Bendat and Piersol [6] and Mark [7], we may define the instantaneous power spectral density of {w(t)} as

$$\Phi_w(f,t) \triangleq \int_{-\infty}^{\infty} \phi_w(\tau,t)e^{-i2\pi f\tau}d\tau \tag{2.2a}$$

$$= \int_{-\infty}^{\infty} \phi_w(\tau,t)\ \cos\ (2\pi f\tau)d\tau \tag{2.2b}$$

$$= 2\int_{0}^{\infty} \phi_w(\tau,t)\ \cos\ (2\pi f\tau)d\tau \quad , \tag{2.2c}$$

where the second and third lines follow from the fact that $\phi_w(\tau,t)$ is a real and even function of τ. From Eq. (2.2b), it is immediately evident that the instantaneous spectrum $\phi_w(f,t)$ is real and is an even function of frequency f.

From the Fourier mate to Eq. (2.2a), we may express $\phi_w(\tau,t)$ in terms of $\Phi_w(f,t)$:

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*In Sec. 4, angular brackets with a subscript x are used to denote a space average over the coordinate x.

$$\phi_W(\tau,t) = \int_{-\infty}^{\infty} \Phi_W(f,t)e^{i2\pi f\tau}df \tag{2.3a}$$

$$= \int_{-\infty}^{\infty} \Phi_W(f,t) \cos (2\pi f\tau)df \tag{2.3b}$$

$$= 2 \int_{0}^{\infty} \Phi_W(f,t) \cos (2\pi f\tau)df \quad , \tag{2.3c}$$

where, again, the second and third lines are a consequence of the fact that $\Phi_W(f,t)$ is real and is an even function of f.

To provide motivation for the term "instantaneous power spectrum", we note that by substituting Eq. (2.1) into the left-hand sides of Eqs. (2.3b) and (2.3c) and setting $\tau = 0$ in the resulting expressions, we have

$$\langle w^2(t)\rangle = \int_{-\infty}^{\infty} \Phi_W(f,t)df \tag{2.4a}$$

$$= 2 \int_{0}^{\infty} \Phi_W(f,t)df \quad . \tag{2.4b}$$

Hence, $\Phi_W(f,t)$ is a frequency decomposition of the expected "instantaneous power" $\langle w^2(t)\rangle$ of the process $\{w(t)\}$. Furthermore, in the case of a stationary process w(t), we have

$$\phi_W(\tau,t) = \langle w(t)\ w(t+\tau)\rangle = R_W(\tau) \quad , \tag{2.5}$$

where $R_W(\tau)$ is the usual definition of the autocorrelation function of a stationary process. It follows directly from Eq. (2.5) and the definition of Eq. (2.2), that the instantaneous spectrum $\Phi_W(f,t)$ is independent of t in the case of stationary processes and that it reduces to the usual definition of the power spectral density in these cases.

In the case of ergodic processes with zero mean value, we have for the usual autocorrelation function

$$R_w(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} w(t-\tfrac{\tau}{2})\; w(t+\tfrac{\tau}{2})dt \tag{2.6a}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi_w(\tau,t)dt \quad , \tag{2.6b}$$

where it is assumed that $\phi_w(\tau,t)$ decays to zero sufficiently fast as a function of τ so that the above limits exist. Notice that, in Eq. (2.6b), we have written

$$\phi_w(\tau,t) = w(t-\tfrac{\tau}{2})\; w(t+\tfrac{\tau}{2}) \quad ; \tag{2.7}$$

i.e., no ensemble average has been taken. Consequently, for an ergodic process, it follows that the usual power spectral density may be obtained from a time average of the instantaneous spectrum; i.e.,

$$\Phi_w(f) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \Phi_w(f,t)dt \quad , \tag{2.8}$$

where we have again assumed that the limit in Eq. (2.8) exists.* In words, for an ergodic random process, the time average of the instantaneous spectrum of a single sample function approaches the usual power spectral density of the process as the averaging time approaches arbitrarily large values. It will become evident later that this property of the instantaneous spectrum is very important in the present work. Other time-dependent spectrum representations [*8*], do not satisfy the property described by Eq. (2.8) for the instantaneous spectrum.

Input-Response Relationships

Consider an aircraft with spatial extent negligible in comparison with the scale of turbulence. We may characterize the aircraft by its response h(t) to a "unit impulse" of turbulence velocity occurring at t = 0. To discuss the input-response relationships for the aircraft, it is necessary to define the instantaneous

*For simplicity, we have ignored the problem of the existence of the limit in Eq. (2.8). See, for example, Davenport and Root [*9*] pp. 107-108, or Middleton [*10*]. Our neglect of this point has no practical significance for the present work.

autocorrelation function of the unit-impulse response function h(t) as

$$\phi_h(\tau,t) \triangleq h(t-\tfrac{\tau}{2})\ h(t+\tfrac{\tau}{2}) \quad , \tag{2.9}$$

from which we may define the instantaneous spectrum of h(t) as

$$\Phi_h(f,t) \triangleq \int_{-\infty}^{\infty} \phi_h(\tau,t)e^{-i2\pi f\tau}d\tau \quad . \tag{2.10}$$

Let $\{y(t)\}$ denote the generally nonstationary response process of the aircraft, and let $\phi_y(\tau,t)$ and $\Phi_y(f,t)$ denote the instantaneous autocorrelation function and instantaneous spectrum of the response. Then, in Mark [*7,11*] it is shown that the instantaneous autocorrelation function and instantaneous spectrum of the response process are related to the corresponding characterizations of the input process and aircraft by

$$\phi_y(\tau,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_h(\xi,u)\ \phi_w(\tau-\xi,t-u)d\xi du \tag{2.11}$$

and

$$\Phi_y(f,t) = \int_{-\infty}^{\infty} \Phi_h(f,u)\ \Phi_w(f,t-u)du \quad . \tag{2.12}$$

Consider the above expression for $\phi_y(\tau,t)$ evaluated at $\tau = 0$. Using the fact that $\phi_w(\xi,t)$ is an even function of ξ, it follows directly from Eq. (2.11) that the instantaneous mean-square aircraft response may be expressed at a generic time $t = t'$ as

$$\langle y^2(t')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_h(\xi,u)\ \phi_w(\xi,t'-u)d\xi du \quad . \tag{2.13}$$

From Eqs. (2.9) and (2.13), it is easy to show that if h(t) is zero everywhere except within the interval $0 \le t \le T$, then the only region of $\phi_w(\xi,t)$ that affects $\langle y^2(t')\rangle$ is the shaded area shown in Fig. 1. Furthermore, one can show that the only interval of t of the process $\{w(t)\}$ that affects the shaded region shown in Fig. 1 is the interval $(t'-T) \le t \le t'$. Consequently, for the purpose of computing the mean-square response of h(t) at time t' by Eq. (2.13), it is necessary to represent the autocorrelation function $\phi_w(\xi,t)$ accurately only within the shaded region shown in the figure. This fact is important, because it implies that we may use *local*

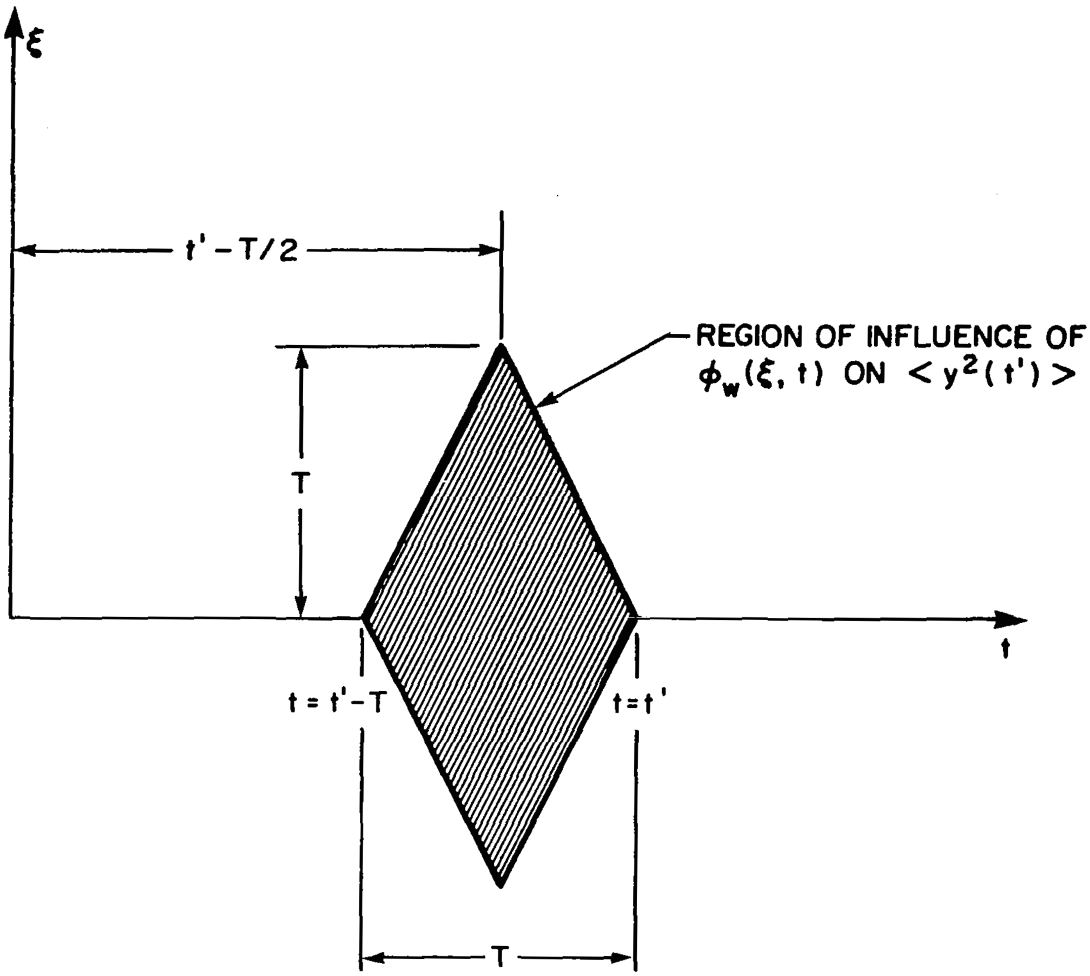

FIG. 1. REGION OF INFLUENCE OF $\phi_w(\xi,t)$ ON $<y^2(t')>$ FOR SYSTEM IMPULSE RESPONSE FUNCTION h(t) THAT IS NONZERO ONLY WITHIN THE INTERVAL $0 < t < T$.

representations of $\phi_w(\xi,t)$, valid only within the diamond shown in Fig. 1, for the purpose of determining the mean-square response of an aircraft.

UNIFORMLY MODULATED MODEL OF NONHOMOGENEOUS TURBULENCE

In the previous section, we discussed a power spectral representation of *nonstationary* turbulence velocities. In the turbulence literature, it is customary to consider the corresponding nonhomogeneous behavior instead, where the nonhomogeneity is with respect to a spatial coordinate, say x. Transformation between the time coordinate t and the spatial coordinate x is carried out by invoking Taylor's hypothesis — i.e., x = Vt — where, in the present application, V is the speed of the aircraft used in making the turbulence measurements. From here on, we shall refer to nonhomogeneous rather than nonstationary behavior.

Definition of Uniformly Modulated Nonhomogeneous Turbulence

One of the simplest forms of nonhomogeneous behavior is nonhomogeneity in the intensity or "instantaneous variance" of a record of turbulence velocities. The effects of such nonhomogeneous behavior on the spectrum of turbulence have been debated for some time. See, for example, Fig. 19 and the accompanying discussion in the review paper "Atmospheric Turbulence" [*1*].

A quantitative model of such nonhomogeneous behavior is the representation of a turbulence velocity record w(x) by the product law

$$w(x) = \sigma(x)\ z(x) \quad , \tag{3.1a}$$

$$\langle z^2 \rangle = 1 \quad , \tag{3.1b}$$

where z(x) is a homogeneous (i.e., stationary) random process with unit standard deviation and $\sigma(x)$ is the nonhomogeneous standard deviation of w(x). Thus, $\sigma(x)$ is necessarily nonnegative. Processes of the form of Eq. (3.1) have been referred to by Shinozuka (1964, 1965) as uniformly modulated.

If we have a single record of nonhomogeneous turbulence w(x) that is assumed to obey the product law of Eq. (3.1) and if $\sigma(x)$ is estimated from this single record, we may regard $\sigma(x)$ either as a deterministic function or as a sample function of a random process. We shall discuss both of these interpretations below. For both, it will be assumed that z(x) is a Gaussian process.

Instantaneous Spectrum of Uniformly Modulated Turbulence

Case of deterministic $\sigma(x)$. The instantaneous spectrum of w(x) is defined as the Fourier transform of the instantaneous autocorrelation function $\phi_x(\xi,x)$; i.e.,

$$\Phi_w(k,x) \triangleq \int_{-\infty}^{\infty} \phi_w(\xi,x)e^{-i2\pi k\xi}d\xi \quad , \tag{3.2}$$

where $\phi_w(\xi,x)$ is defined as

$$\phi_w(\xi,x) \triangleq \langle w(x-\tfrac{\xi}{2})\ w(x+\tfrac{\xi}{2})\rangle \quad . \tag{3.3}$$

Substituting the product model of Eq. (3.1) into Eq. (3.3) and first considering $\sigma(x)$ to be deterministic, we have for the instantaneous autocorrelation function of w(x),

$$\phi_w(\xi,x) = \sigma(x-\tfrac{\xi}{2})\ \sigma(x+\tfrac{\xi}{2})\ \langle z(x-\tfrac{\xi}{2})\ z(x+\tfrac{\xi}{2})\rangle \tag{3.4a}$$

$$= \phi_\sigma(\xi,x)\ \phi_z(\xi) \quad , \tag{3.4b}$$

where, here, we have defined for deterministic $\sigma(x)$,

$$\phi_\sigma(\xi,x) \triangleq \sigma(x-\tfrac{\xi}{2})\ \sigma(x+\tfrac{\xi}{2}) \tag{3.5}$$

and

$$\phi_z(\xi) \triangleq \langle z(x-\tfrac{\xi}{2})\ z(x+\tfrac{\xi}{2})\rangle \tag{3.6a}$$

$$= \langle z(x)\ z(x+\xi)\rangle \tag{3.6b}$$

since z(x) is, by assumption, homogeneous. From Eqs. (3.1b) and (3.6), it follows that

$$\phi_z(0) \equiv 1 \quad . \tag{3.7}$$

Substituting Eq. (3.4b) into Eq. (3.2), it follows that

$$\Phi_w(k,x) = \int_{-\infty}^{\infty} \phi_\sigma(\xi,x)\ \phi_z(\xi)e^{-i2\pi k\xi}d\xi \quad . \tag{3.8}$$

Applying the convolution theorem to Eq. (3.8), we may express $\Phi_w(k,x)$ in terms of the Fourier transforms $\Phi_\sigma(k,x)$ and $\Phi_z(k)$ of $\phi_\sigma(\xi,x)$ and $\phi_z(\xi)$; i.e.,

$$\Phi_w(k,x) = \int_{-\infty}^{\infty} \Phi_\sigma(\nu,x)\ \Phi_z(k-\nu)d\nu \quad , \tag{3.9}$$

where the definitions of $\Phi_\sigma(k,x)$ and $\Phi_z(k)$ are

$$\Phi_\sigma(k,x) \triangleq \int_{-\infty}^{\infty} \phi_\sigma(\xi,x)e^{-i2\pi k\xi}d\xi \quad , \tag{3.10}$$

and

$$\Phi_z(k) \triangleq \int_{-\infty}^{\infty} \phi_z(\xi)e^{-i2\pi k\xi}d\xi \quad . \tag{3.11}$$

Equation (3.9) is a fundamental relationship that expresses the instantaneous wavenumber spectrum $\Phi_w(k,x)$ of the process $w(x)$ as the wavenumber convolution of the instantaneous wavenumber spectrum $\Phi_\sigma(k,x)$ of the nonhomogeneous standard deviation $\sigma(x)$ and the usual wavenumber spectrum $\Phi_z(k)$ of the homogeneous process $z(x)$.

Case of possibly nonhomogeneous stochastic process $\sigma(x)$. Equations (3.6) to (3.11) also apply to situations where $\sigma(x)$ is considered to be a possibly nonhomogeneous stochastic process that is statistically independent of the process $z(x)$, except that in these cases, the instantaneous autocorrelation function of $\sigma(x)$ is defined as the mathematical expectation of the right-hand side of Eq. (3.5); i.e.,

$$\phi_\sigma(\xi,x) = \langle\sigma(x-\tfrac{\xi}{2})\ \sigma(x+\tfrac{\xi}{2})\rangle \quad . \tag{3.12}$$

Case of homogeneous process $\sigma(x)$. When the process $\sigma(x)$ is homogeneous, the instantaneous autocorrelation function of $\sigma(x)$ becomes independent of x; thus, for homogeneous $\sigma(x)$, we have

$$\begin{aligned}\phi_\sigma(\xi,x) = \phi_\sigma(\xi) &= \langle\sigma(x-\tfrac{\xi}{2})\ \sigma(x+\tfrac{\xi}{2})\rangle \\ &= \langle\sigma(x)\ \sigma(x+\xi)\rangle \quad .\end{aligned} \tag{3.13}$$

Thus, for homogeneous processes $\sigma(x)$, it follows from Eq. (3.10) that $\Phi_\sigma(k,x)$ reduces to the usual wavenumber spectrum of $\sigma(x)$, which is

$$\Phi_\sigma(k,x) = \Phi_\sigma(k) = \int_{-\infty}^{\infty} \phi_\sigma(\xi) e^{-i2\pi k\xi} d\xi \quad . \tag{3.14}$$

Hence, when $\sigma(x)$ is homogeneous and independent of $z(x)$ it follows from Eq. (3.9) that $\Phi_W(k,x)$ is independent of x and is expressed in terms of the spectra of $\sigma(x)$ and $z(x)$ by the convolution of $\Phi_\sigma(k)$ and $\Phi_z(k)$:

$$\Phi_W(k) = \int_{-\infty}^{\infty} \Phi_\sigma(\nu) \; \Phi_z(k-\nu) d\nu \quad . \tag{3.15}$$

Case of ergodic process $\sigma(x)$. When $\sigma(x)$ is an ergodic process, we may obtain the autocorrelation function of $\sigma(x)$ from a single realization of $\sigma(x)$; i.e.,

$$\phi_\sigma(\xi) = \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} \sigma(x-\tfrac{\xi}{2}) \; \sigma(x+\tfrac{\xi}{2}) dx \quad . \tag{3.16}$$

Equations (3.15) and (3.14) apply to ergodic processes $\sigma(x)$ that are independent of $z(x)$, where, in these cases, the autocorrelation function of $\sigma(x)$ may be evaluated by using Eq. (3.16).

For ergodic processes $\sigma(x)$ that are independent of $z(x)$, we may also determine the wavenumber spectrum of $w(x)$ by taking a "space average" of Eq. (3.9), where, in this case, we interpret $\Phi_\sigma(k,x)$ as having been obtained from Eqs. (3.10) and (3.5); thus,

$$\Phi_W(k) = \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} \Phi_W(k,x) dx \tag{3.17a}$$

$$= \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} \int_{-\infty}^{\infty} \Phi_\sigma(\nu,x) \; \Phi_z(k-\nu) d\nu dx$$

$$= \int_{-\infty}^{\infty} \left[\lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} \Phi_\sigma(\nu,x) dx \right] \Phi_z(k-\nu) d\nu$$

$$= \int_{-\infty}^{\infty} \Phi_\sigma(\nu) \; \Phi_z(k-\nu) d\nu \quad , \tag{3.17b}$$

where we have obtained $\Phi_\sigma(\nu)$ above by the "space average"

$$\Phi_\sigma(\nu) = \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} \Phi_\sigma(\nu,x)dx \tag{3.18a}$$

$$= \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} \int_{-\infty}^{\infty} \phi_\sigma(\xi,x)e^{-i2\pi\nu\xi}d\xi dx$$

$$= \int_{-\infty}^{\infty} \left[\lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} \phi_\sigma(\xi,x)dx\right] e^{-i2\pi\nu\xi}d\xi$$

$$= \int_{-\infty}^{\infty} \phi_\sigma(\xi)e^{-i2\pi\nu\xi}d\xi \quad , \tag{3.18b}$$

where, here, we interpret $\phi_\sigma(\xi)$ as the space average

$$\phi_\sigma(\xi) = \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} \sigma(x-\tfrac{\xi}{2})\ \sigma(x+\tfrac{\xi}{2})dx \quad , \tag{3.19}$$

as in Eq. (3.16).

Comparison with evolutionary spectrum. The above apparently trivial operations for homogeneous and ergodic processes $\sigma(x)$ have important implications for other definitions of time-dependent spectra. In essence, what we have shown above is that for processes that are homogeneous or ergodic, results that are obtained using the instantaneous spectrum reduce, automatically, to the usually homogeneous or ergodic process results. Specifically, for both homogeneous and ergodic processes, Eq. (3.15) [or Eq. (3.17b)] provides an expression for the "power" spectrum of the uniformly modulated process $w(x)$, where, for homogeneous processes, $\Phi_\sigma(k)$ is the Fourier transform of the autocorrelation function obtained by the ensemble average, Eq. (3.13); whereas, the ergodic processes $\Phi_\sigma(k)$ may be interpreted either as the "space average" of the instantaneous spectrum defined for a single realization of $\sigma(x)$ with no ensemble average [e.g., Eq. (3.18a)] or as the Fourier transform of the autocorrelation function of $\sigma(x)$ obtained by a "space average" as in Eqs. (3.18b) and (3.19).

The evolutionary spectral density of Priestley [*12,13*] does not, generally, satisfy properties comparable to those described above. For example, Howell and Lin [*8*] have used the uniformly modulated model of turbulence [Eq. (1) of their paper] to study the response of flight vehicles to nonstationary atmospheric turbulence. Their approach uses the evolutionary spectral density. It immediately follows from Eqs. (1), (3), and (A4), and the comment immediately following Eq. (2) of their paper, that for processes $\sigma(x)$ that are either homogeneous or ergodic [processes c(t) that are stationary in their notation], the power spectrum of w(x) is proportional to $\Phi_z(k)$. [In their notation, the power spectrum of W(t) is proportional to $\Phi_{GG}(\omega)$.] It must be concluded that their spectral representation fails to satisfy an important consistency requirement; namely, it does not reduce to the correct result in cases where the component processes are taken to be either stationary or ergodic. That is, in cases where $\sigma(x)$ and z(x) are homogeneous (stationary) or ergodic, the spectrum of w(x) is the convolution of the spectra of $\sigma(x)$ and z(x), as is indicated by our Eq. (3.15).

Measurement of Spectrum of Homogeneous Component Using the Arcsin Law

In order to evaluate the contribution to the spectrum of w(x) from the homogeneous component z(x), it is evident, from Eq. (3.9) [which applies to the case of deterministic $\sigma(x)$] and from Eqs. (3.15) and (3.17) [which apply respectively to homogeneous and ergodic stochastic processes $\sigma(x)$], that one must evaluate the spectrum $\Phi_z(k)$ of the homogeneous process z(x). A very convenient feature of the uniformly modulated model of nonhomogeneous turbulence of Eq. (3.1) is that when z(x) is assumed to be a Gaussian random process, we have at our disposal a method to evaluate the spectrum of z(x) from one or more recordings of the process w(x). To recognize this, we first note that since $\sigma(x)$ is, by definition, nonnegative, the zero crossings of the processes w(x) and z(x) in Eq. (3.1a) necessarily coincide; moreover, the sign (positive or negative) of the processes w(x) and z(x) between zero crossings also coincide. Thus, for any recording w(x) that satisfies the product law of Eq. (3.1a), we may evaluate both the positions of the zero crossings and the signs between zero crossings of the process z(x) from these same quantities evaluated from the recording w(x). Furthermore, it is known that from the positions of the zero crossings and the signs of a Gaussian random process with zero mean value, one can evaluate the power spectrum of the process using the so-called arcsin law; see, for example, Lawson and Uhlenbeck [5], pp. 57-58.

In the present application, the arcsin law may be described as follows. Compute the mean value of a recording of the process w(x). Form a new sample from w(x) by subtracting the mean value from the original recording. "Infinitely clip" this new sample so that only the positions of the zero crossings and the signs of the process between zeros are retained. Then construct a new sample function that has a value of +1 everywhere where the clipped sample was positive and -1 everywhere where the clipped sample was negative. Construct the autocorrelation function of this newly constructed sample function and call it $\phi_0(\xi)$. Call the autocorrelation function of the original sample function (with mean value subtracted out) $\phi_z(\xi)$. Then, if the sample functions are of sufficient length to provide good statistical reliability, the arcsin law states that we may reconstruct $\phi_z(\xi)$ from $\phi_0(\xi)$ by

$$\phi_z(\xi) = \sin \left[\frac{\pi}{2} \phi_0(\xi)\right] \quad . \tag{3.20}$$

By Fourier transforming both sides of Eq. (3.20), we obtain a relationship for the spectrum of z(x) in terms of properties of the (infinitely clipped) waveform w(x).

Verification of the Arcsin Law Using a Homogeneous Turbulence Record

Cumulative probability distribution function of stationary record. Before using the sine transformation of Eq. (3.20) with nonhomogeneous records, we checked the validity of Eq. (3.20) directly, using a record of turbulence that appeared to be homogeneous — i.e., stationary. The record chosen for this test was the vertical component of Test No. 190, Leg No. 5 of the LO-LOCAT program [*14*]. This velocity record is shown in Fig. 2 of this report.

First, the mean value, standard deviation, and cumulative probability distribution function of this 27,000 sample point record were computed. The mean value and the standard deviation computed from the vertical component of Test No. 190, Leg No. 5 are

$$m = -.039 \text{ m/sec } (-0.128 \text{ ft/sec}) \tag{3.21}$$

$$\sigma = 2.94 \text{ m/sec } (9.63 \text{ ft/sec}) \quad . \tag{3.22}$$

The cumulative probability distribution function P(W) is defined as

$$P(W) \triangleq \int_{-\infty}^{W} p(w)dw \quad , \tag{3.23}$$

where p(w) is the (empirical) probability density function of the record. P(W) was computed for increments of W of .03 m/sec (0.1 ft/sec). The computed values of P(W), which are tabulated in Appendix A of this report, are plotted on the Gaussian probability coordinates in Fig. 3 as discrete points. The solid straight line

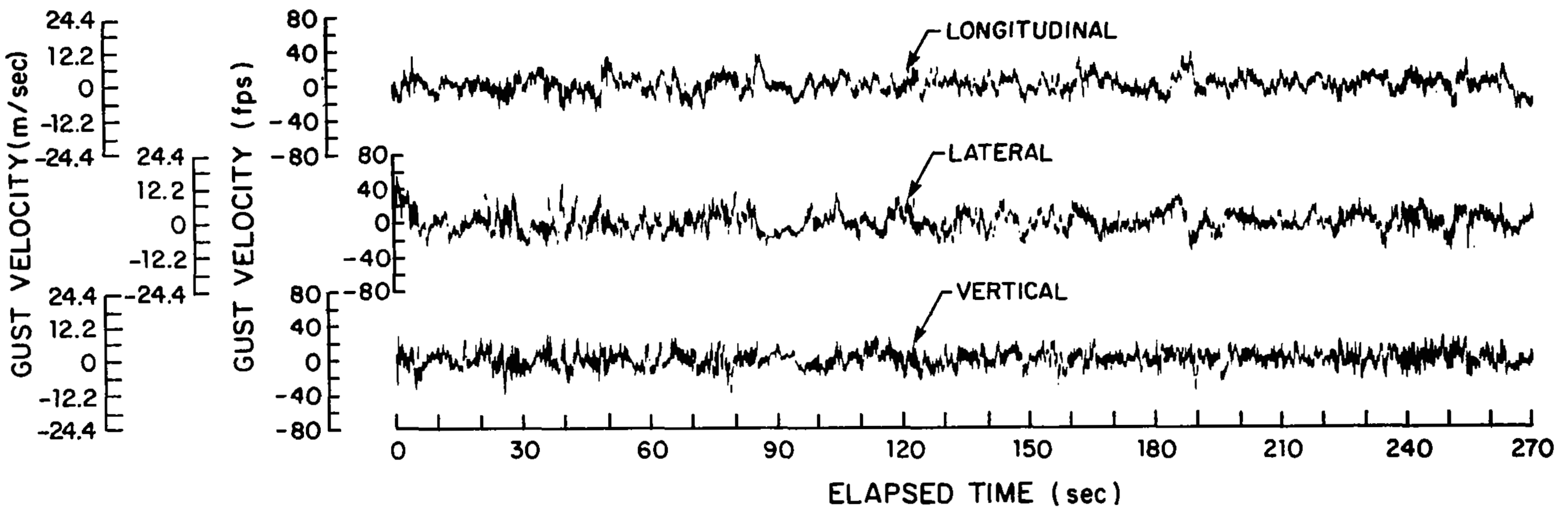

FIG. 2. TURBULENCE RECORDS FROM TEST NO. 190, LEG NO. 5 OF THE LO-LOCAT PROGRAM. (From Jones *et al*, p. 223)

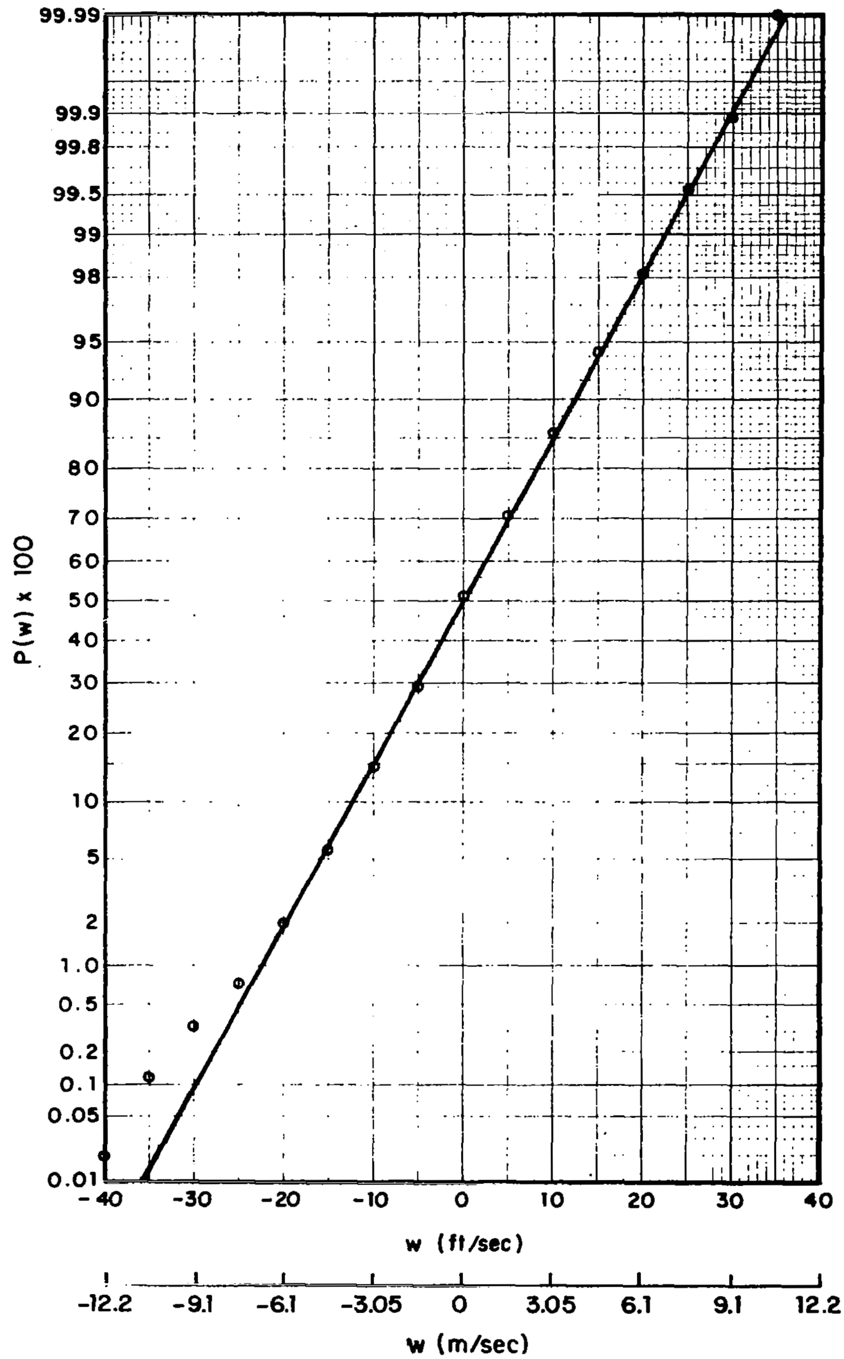

FIG. 3. CUMULATIVE PROBABILITY DISTRIBUTION FUNCTION OF VERTICAL VELOCITY COMPONENT OF TEST NO. 190, LEG NO. 5 OF LO-LOCAT PROGRAM (ENCIRCLED POINTS) COMPARED WITH GAUSSIAN DISTRIBUTION FUNCTION (SOLID LINE) WITH SAME MEAN VALUE (m = -0.039 m/sec, -0.128 ft/sec) AND STANDARD DEVIATION (σ = 2.94 m/sec, 9.63 ft/sec).

shown in Fig. 3 represents a Gaussian cumulative probability distribution function with the same values of mean and standard deviation as those given by Eqs. (3.21) and (3.22). It is evident from Fig. 3 that little significant difference exists between the empirically determined distribution and the Gaussian distribution function. Consequently, insofar as its first order properties are concerned, the vertical component of Test No. 190, Leg No. 5 appears to be very nearly Gaussian. This is helpful because the sine transformation of Eq. (3.20) assumes that the process z(x) is Gaussian.

Conventional power spectral density of stationary record. For a direct test of the validity of the sine transformation of Eq. (3.20), it was necessary first to compute the power spectral density of the record in the conventional way. The method used is described in Appendix B. A Papoulis window function was used to carry out the smoothing of the spectrum in order to get adequate statistical reliability. This window function has a minimum bias property [*15*] and, in this sense, is optimum. The smoothed spectrum computed in the conventional way is shown in Fig. 4.

Power spectral density computed from infinitely clipped stationary record and corrected using the arcsin law. To compute the power spectral density of the record shown in Fig. 2 using the sine transformation of Eq. (3.20), we first removed the mean value of the record, m = -0.039 m/sec (-0.128 ft/sec). At every discrete sample point of the record, the sign of the waveform was determined, and a new record was generated with values of +1 where the original record was positive and of -1 where the original record was negative. The (unsmoothed) power spectrum of this "clipped record" was then computed and its (inverse) Fourier transform was taken, yielding the autocorrelation function $\phi_0(z)$ of the clipped record. A corrected autocorrelation function $\phi_z(\xi)$ was then computed from $\phi_0(z)$, using the sine transformation of Eq. (3.20). This corrected autocorrelation function was then multiplied by the Fourier transform of the same Papoulis (frequency) window function used in the conventional spectrum computation, and the Fourier transform of this product was taken. This final result is the smoothed power spectral density computed using the "arcsin law", as shown in Fig. 4. The details of the above computational procedure are described in Appendix B.

It is evident from Fig. 4 that the differences between the spectrum computed using the conventional method and that computed using "infinite clipping" are insignificant. We must therefore conclude that the stationary record shown in Fig. 2 is sufficiently close to a stationary Gaussian random function for the sine transformation of Eq. (3.20) to be considered valid.

Comparison of spectrum of stationary record with the smoothed von Karman spectrum. Since, for statistical reliability, the spectra computed from the vertical component of the stationary record of Fig. 2 were frequency-smoothed, the von Karman (vertical component)

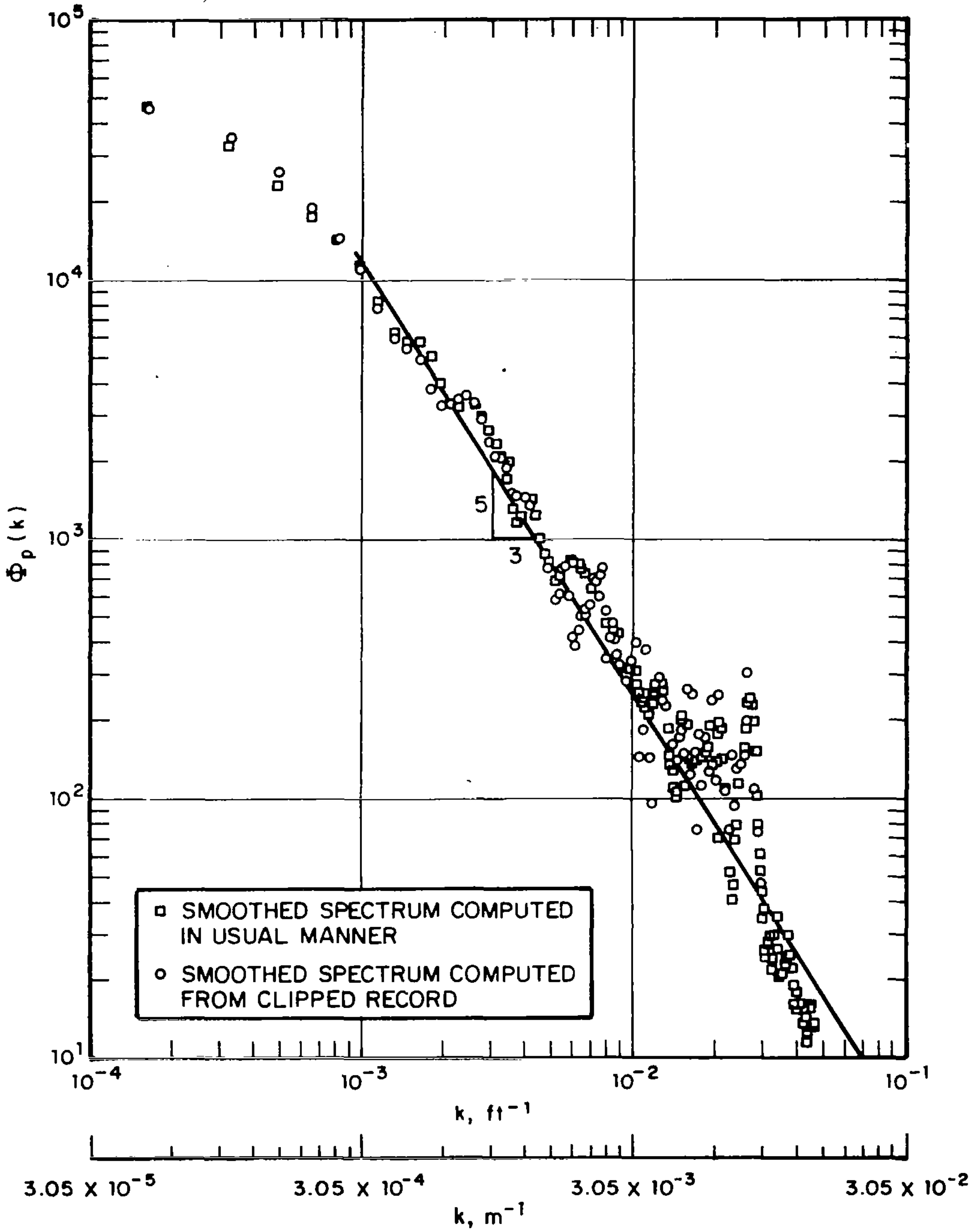

FIG. 4. COMPARISON OF SPECTRA COMPUTED FROM HOMOGENEOUS VERTICAL COMPONENT SHOWN IN FIG. 2. SPECTRA WERE COMPUTED IN THE CONVENTIONAL WAY AND BY INFINITE CLIPPING AFTER CORRECTION BY THE "ARCSIN LAW." BOTH SPECTRA SMOOTHED BY THE SAME PAPOULIS WINDOW FUNCTION.

spectrum should be smoothed by the same frequency window before comparison with the above-described empirically determined spectra. A close fit of the empirical and smoothed von Karman spectra will then indicate that the stationary record may be regarded as having a von Karman spectrum. A description of the smoothing procedure for the von Karman spectrum is described in Appendix C.

Smoothed von Karman spectra evaluated with a standard deviation of σ = 2.94 m/sec (9.63 ft/sec) and with the integral scales of L = 91.44, 121.92, 152.4, 182.88 and 213.36 m (300, 400, 500, 600, and 700 ft) are plotted in Fig. 5 along with the spectrum computed using clipping and the sine transformation. It is evident from Fig. 5 that the von Karman spectrum with an integral scale of about L = 137.16 m (450 ft) provides a reasonable fit to the empirically determined spectrum.

Statistical Confidence of Nonhomogeneous Variance Estimates

The most straightforward procedure for estimating the nonhomogeneous standard deviation $\sigma(x)$ in the model of Eq. (3.1) is to square and form a local average of a sample function w(x):

$$\hat{\sigma}^2(x) = \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} \sigma^2(x)\ z^2(x)dx \qquad (3.24)$$

Taking the expected value of the above equation, we obtained upon using Eq. (3.1b):

$$<\hat{\sigma}^2(x)> = \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} \sigma^2(x)\ <z^2(x)>dx$$

$$= \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} \sigma^2(x)dx \qquad (3.25)$$

Let Var $\{\hat{\sigma}^2\}$ denote the variance of the estimate of $\sigma^2(x)$ given by Eq. (3.25). Then, in Appendix D, it is shown that the ratio of the variance of our estimate $\hat{\sigma}^2(x)$ to the square of the expected value of $\sigma^2(x)$, is given approximately by

$$\frac{\text{Var } \{\hat{\sigma}^2\}}{m^2\ \{\hat{\sigma}^2\}} \approx \frac{1}{\Delta k \Delta x}$$

$$= 1.069 \frac{L}{\Delta x} , \qquad (3.26)$$

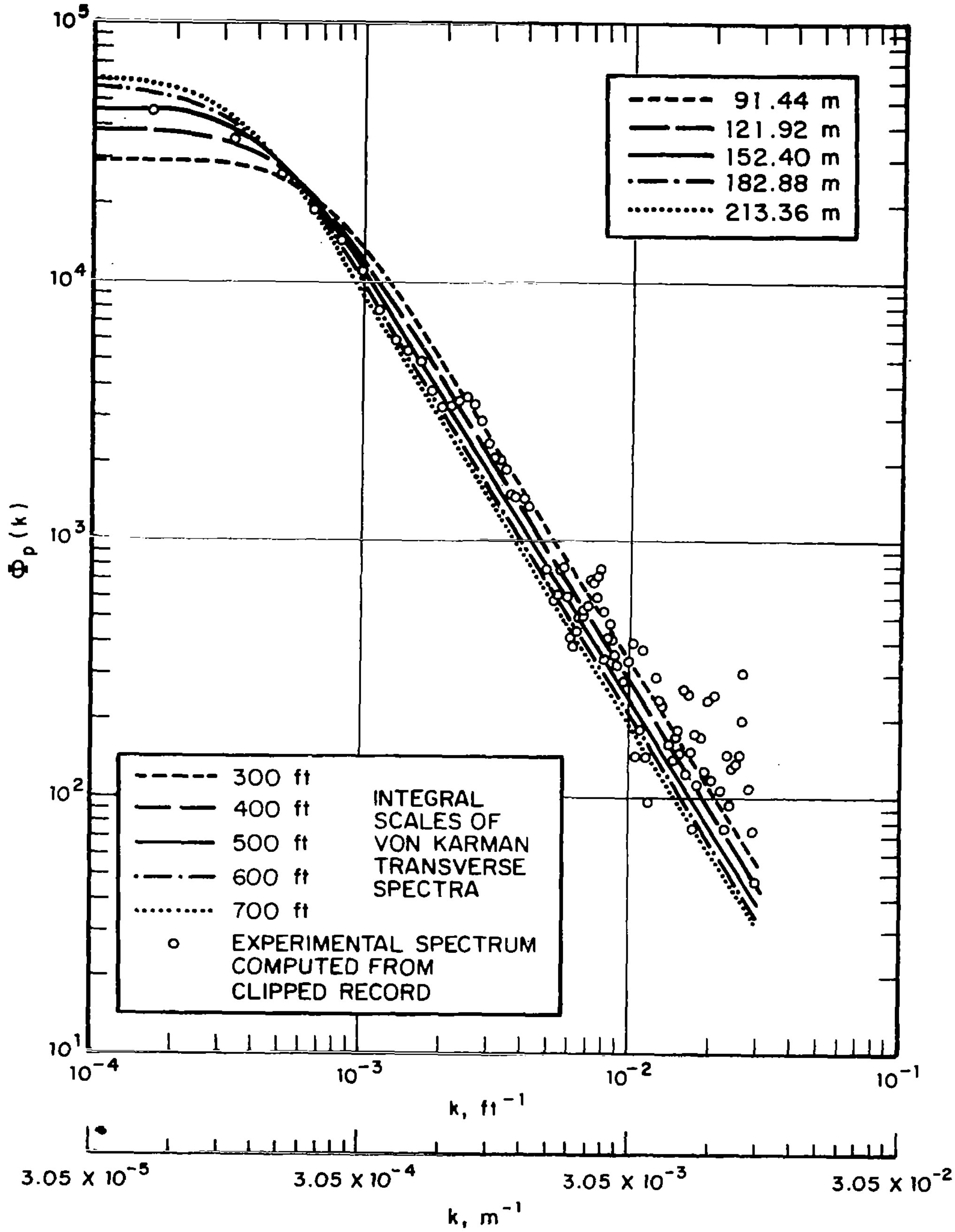

FIG. 5. COMPARISON OF SMOOTHED von KARMAN TRANSVERSE SPECTRA σ = 2.94 m/sec (9.63 ft/sec) WITH THE SPECTRUM OF THE VERTICAL RECORD SHOWN IN FIG. 2 COMPUTED USING "INFINITE CLIPPING" AND THE "ARCSIN LAW." ALL SPECTRA WERE SMOOTHED BY THE SAME PAPOULIS WINDOW FUNCTION.

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SERIES EXPANSION OF INSTANTANEOUS SPECTRUM OF UNIFORMLY MODULATED TURBULENCE

Derivation of Series Expansion

According to Eq. (3.4b), the instantaneous autocorrelation function $\phi_w(\xi,x)$ of uniformly modulated turbulence may be expressed as

$$\phi_w(\xi,x) = \phi_\sigma(\xi,x)\ \phi_z(\xi) \quad , \tag{4.1}$$

where $\phi_\sigma(\xi,x)$ is defined by Eq. (3.5) for deterministic modulating functions $\sigma(x)$, by Eq. (3.12) for possibly nonhomogeneous, stochastic $\sigma(x)$, and by Eq. (3.16) for ergodic $\sigma(x)$. Here, we are interested in determining the "threshold" in the rate of fluctuation of $\sigma(x)$ that begins to have an effect on the spectrum of $w(x)$. Thus, we are interested in modulating functions $\sigma(x)$ that vary slowly in comparison with the fluctuations in the stationary component $z(x)$. [See Eq. (3.1a) for review of components.] This behavior will be manifested in the appearance of the instantaneous autocorrelation functions $\phi_\sigma(\xi,x)$ and $\phi_z(\xi)$ in that $\phi_\sigma(\xi,x)$ will change gradually about the point $\xi = 0$ in comparison with $\phi_z(\xi)$. Such behavior is illustrated in Fig. 6.

The behavior illustrated by Fig. 6 suggests that it should be possible to represent $\phi_\sigma(\xi,x)$ by a few terms in its Maclaurin expansion in the variable ξ over the range of ξ where $\phi_z(\xi)$ is not negligible. To include cases where the derivatives of $\phi_\sigma(\xi,x)$ are not continuous at $\xi = 0$, we shall initially consider the Maclaurin expansion of $\phi_\sigma(\xi,x)$ valid in the region $\xi \geq 0$ by using right-hand derivatives of $\phi_\sigma(\xi,x)$ evaluated at the origin, $\xi = 0$. We denote these right-hand derivatives of $\phi_\sigma(\xi,x)$ with respect to ξ by

$$\phi_\sigma^{(n)}(0+,x) \triangleq \left.\frac{\partial^n \phi_\sigma(\xi,x)}{\partial \xi^n}\right|_{\xi\ =\ 0+} \quad . \tag{4.2}$$

Then, the Maclaurin expansion of $\phi_\sigma(\xi,x)$, valid for positive and negative ξ, may be expressed as

$$\phi_\sigma(\xi,x) = \sum_{n=0}^{N} \frac{\phi_\sigma^{(n)}(0+,x)}{n!}\ |\xi|^n + r_{N+1}(\xi,x) \quad , \tag{4.3}$$

where

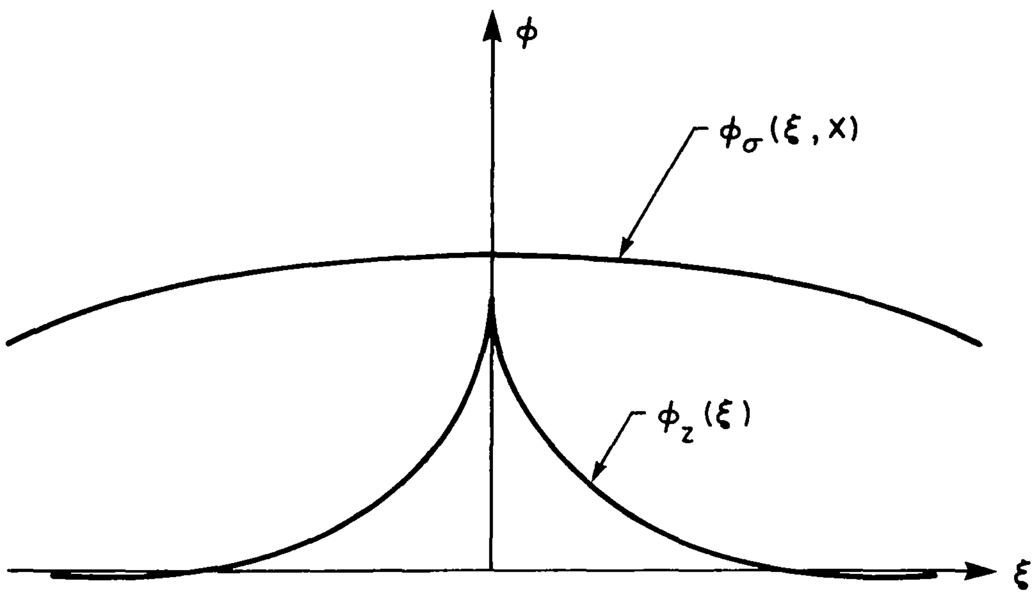

FIG. 6. BEHAVIOR OF $\phi_\sigma(\xi,x)$ AND $\phi_z(\xi)$ NEAR $\xi = 0$ FOR CASE WHERE MODULATION $\sigma(x)$ IS SLOW IN COMPARISON WITH FLUCTUATIONS IN $z(x)$.

$$r_{N+1}(\xi,x) = \frac{1}{N!}\int_0^{|\xi|} \phi_\sigma^{(N+1)}(\eta,x)\ (|\xi| - \eta)^N\, d\eta \tag{4.4}$$

$$= \frac{\phi_\sigma^{(N+1)}(\eta,x)}{(N+1)!}\,|\xi|^{N+1}\ , \qquad 0 < \eta < \infty \tag{4.5}$$

where in the second form, Eq. (4.5), the (N+1)*st* derivative $\phi_\sigma^{(N+1)}(\eta,x)$ is to be evaluated at some (generally unknown) point within $0 < \eta < \infty$. The absolute value signs in Eqs. (4.3) to (4.5) are a consequence of the fact that $\phi_\sigma(\xi,x)$ must be an even function of ξ.

Let us now use Eq. (3.8) to obtain an expression for $\Phi_w(k,x)$ using the above expansion. Substituting Eq. (4.3) into Eq. (3.8) and interchanging orders of summation and integration gives

$$\Phi_w(k,x) = \sum_{n=0}^{N} \frac{\phi_\sigma^{(n)}(0+,x)}{n!}\int_{-\infty}^{\infty} |\xi|^n\ \phi_z(\xi)e^{-i2\pi k\xi}d\xi + R_{N+1}(k,x) \tag{4.6a}$$

$$= 2\sum_{n=0}^{N} \frac{\phi_\sigma^{(n)}(0+,x)}{n!}\int_0^{\infty} \xi^n\ \phi_z(\xi)\ \cos\ 2\pi k\xi d\xi + R_{N+1}(k,x)\ , \tag{4.6b}$$

where

$$R_{N+1}(k,x) = \int_{-\infty}^{\infty} r_{N+1}(\xi,x)\ \phi_z(\xi)e^{-i2\pi k\xi}d\xi \tag{4.7a}$$

$$= 2\int_0^{\infty} r_{N+1}(\xi,x)\ \phi_z(\xi)\ \cos\ 2\pi k\xi d\xi\ , \tag{4.7b}$$

where we have used the fact that $\phi_z(\xi)$ and $r_{N+1}(\xi,x)$ are always even functions of ξ.

We are particularly interested in cases where the modulating function $\sigma(x)$ is relatively well behaved. If the first N derivatives of $\sigma(x)$ exist, then the first N derivatives $\phi_\sigma^{(n)}(\xi,x)$ are continuous in the variable ξ at $\xi = 0$.

In these cases, the terms n = odd integer in Eqs. (4.6a,b) vanish, and we may rewrite Eq. (4.6a) as

$$\Phi_w(k,x) = \sum_{n=0}^{N} \frac{\phi_\sigma^{(n)}(0,x)}{n!} \int_{-\infty}^{\infty} \xi^n \, \phi_z(\xi) e^{-i2\pi k\xi} d\xi + R_{N+1}(k,x) \quad . \tag{4.8}$$

However, differentiating* Eq. (3.11) n times with respect to k and defining

$$\Phi_z^{(n)}(k) \triangleq \frac{d^n}{dk^n} \Phi_z(k) \quad , \tag{4.9}$$

we have

$$\Phi_z^{(n)}(k) = (-i2\pi)^n \int_{-\infty}^{\infty} \xi^n \, \phi_z(\xi) e^{-i2\pi k\xi} d\xi \quad ; \tag{4.10}$$

hence, Eq. (4.8) may be expressed as

$$\Phi_w(k,x) = \sum_{n=0}^{N} \frac{a_n(x)}{n!} \Phi_z^{(n)}(k) + R_{N+1}(k,x) \quad , \tag{4.11}$$

where

$$a_n(x) \triangleq \frac{\phi_\sigma^{(n)}(0,x)}{(-i2\pi)^n} = \frac{1}{(-i2\pi)^n} \left. \frac{\partial^n \phi_\sigma(\xi,x)}{\partial \xi^n} \right|_{\xi = 0} \tag{4.12}$$

and where we have left the plus sign out of the argument of $\phi_\sigma^{(n)}(0+,x)$ in Eqs. (4.8) and (4.12) because continuity of the first N derivatives of $\sigma(x)$ implies continuity at $\xi = 0$ of the first N derivatives with respect to ξ of $\phi_\sigma(\xi,x)$.

Equation (4.11) is the desired result. The case of the most interest is that where the power spectrum of the component z(x) has the von Karman form. In these cases, we may differentiate $\Phi_z(k)$ N times to obtain the expansion functions $\Phi_z^{(n)}(k)$, n = 0,1,···, N in Eq. (4.11). Evaluation of the expansion coefficients $a_n(x)$ will be discussed shortly.

---

*Differentiation under the integral sign is discussed on p. 443 of Apostol [*16*].

In the applications of the series expansion of Eq. (4.11) that follow, we shall deal mostly with cases where the remainder term $R_{N+1}(k,x)$ is identically zero. Nevertheless, for other situations, bounds can be put on the magnitude of this term as follows. It is evident directly from Eq. (4.7b) that

$$
\begin{aligned}
|R_{N+1}(k,x)| &= 2\left|\int_0^\infty r_{N+1}(\xi,x)\ \phi_z(\xi)\ \cos\ 2\pi k\xi d\xi\right| \\
&\leq 2\int_0^\infty |r_{N+1}(\xi,x)\ \phi_z(\xi)\ \cos\ 2\pi k\xi|d\xi \\
&\leq 2\int_0^\infty |r_{N+1}(\xi,x)||\phi_z(\xi)|d\xi
\end{aligned}
\tag{4.13}
$$

Hence, from Eqs. (4.5) and (4.13) it follows that

$$
|R_{N+1}(k,x)| \leq \max_{0<\eta<\infty} \left|\phi_\sigma^{(N+1)}(\eta,x)\right| \frac{2}{(N+1)!}\int_0^\infty \xi^{N+1}\ |\phi_z(\xi)|d\xi \quad , \tag{4.14}
$$

where $\max_{0<\eta<\infty} \left|\phi_\sigma^{(N+1)}(\eta,x)\right|$ denotes the maximum value of $\left|\phi_\sigma^{(N+1)}(\eta,x)\right|$ within the interval $0<\eta<\infty$. For applications where the form of the autocorrelation functions $\phi_\sigma(\xi,x)$ and $\phi_z(\xi)$ are analytical, Eq. (4.14) may be easily applied to provide a bound on $R_{N+1}(k,x)$. Notice that the bound provided by the right-hand side is independent of wavenumber k.

Derivation of Expressions for Expansion Coefficients

For situations where the instantaneous autocorrelation function $\phi_\sigma(\xi,x)$ is known, Eq. (4.12) provides an expression for evaluating the expansion coefficients $a_n(x)$. Other expressions will now be derived for situations where $\sigma(x)$ is regarded as either a deterministic function or a sample function from a possibly nonhomogeneous process or an ergodic process.

Case of deterministic $\sigma(x)$. Applying Leibniz's formula for the *nth* derivative of the product of two functions to the expression for $\phi_\sigma(\xi,x)$ given by Eq. (3.5), we have

$$\frac{\partial^n \phi_\sigma(\xi,x)}{\partial \xi^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k \sigma(x-\frac{\xi}{2})}{\partial \xi^k} \frac{\partial^{n-k}\sigma(x+\frac{\xi}{2})}{\partial \xi^{n-k}} , \tag{4.15}$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} , \tag{4.16}$$

are the binomial coefficients. But

$$\frac{\partial^k \sigma(x-\frac{\xi}{2})}{\partial \xi^k} = \frac{1}{(-2)^k} \left. \frac{d^k \sigma(\eta)}{d\eta^k} \right|_{\eta=x-\frac{\xi}{2}} \tag{4.17a}$$

and

$$\frac{\partial^{n-k} \sigma(x+\frac{\xi}{2})}{\partial \xi^{n-k}} = \frac{1}{2^{n-k}} \left. \frac{d^{n-k} \sigma(\eta)}{d\eta^{n-k}} \right|_{\eta=x+\frac{\xi}{2}} . \tag{4.17b}$$

Combining Eqs. (4.15) to (4.17) gives

$$\frac{\partial^n \phi_\sigma(\xi,x)}{\partial \xi^n} = \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left. \frac{d^k \sigma(\eta)}{d\eta^k} \right|_{\eta=x-\frac{\xi}{2}} \left. \frac{d^{n-k} \sigma(\eta)}{d\eta^{n-k}} \right|_{\eta=x+\frac{\xi}{2}} . \tag{4.18}$$

Thus, evaluating Eq. (4.18) at $\xi = 0$ and combining the resulting expression with Eq. (4.12) gives

$$a_n(x) = \frac{1}{(-i4\pi)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{d^k \sigma(x)}{dx^k} \frac{d^{n-k}\sigma(x)}{dx^{n-k}} . \tag{4.19}$$

From Eq. (4.19), we shall now show that

$$a_n(x) = 0 , \qquad n = \text{odd} . \tag{4.20}$$

We first note from the definition of the binomial coefficients, Eq. (4.16), that

$$\binom{n}{k} = \binom{n}{n-k} \quad . \tag{4.21}$$

Furthermore, when n is odd, there are always an even number of terms in the summation on the right-hand side of Eq. (4.19). For every term for which $k < (n/2)$, there is a corresponding term $n-k$, where $(n-k) > (n/2)$, which from Eqs. (4.19) and (4.21) has identical magnitude but opposite sign to the term where $k < (n/2)$ whenever n is odd. Consequently, $a_n(x)$ is identically zero when n is odd.

When n is even, there is an odd number of terms in the sum on the right-hand side of Eq. 4.19. The middle term is the term for which $k = n/2$. For every term for which $k < (n/2)$ there is, in this case, a corresponding term $(n-k) > (n/2)$ identical in both magnitude and sign to the corresponding term for which $k < (n/2)$. Hence, noting that $(-1)^n = 1$ when n is even, Eq. (4.19) can be expressed, whenever n is even, as

$$a_n(x) = \frac{1}{(i4\pi)^n} \left\{ 2 \left[\sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n}{k} \frac{d^k\sigma(x)}{dx^k} \frac{d^{n-k}\sigma(x)}{dx^{n-k}} \right] + (-1)^{n/2} \binom{n}{n/2} \left[\frac{d^{n/2}\sigma(x)}{dx^{n/2}} \right]^2 \right\} , \quad n = \text{even} \quad . \tag{4.22}$$

Equations (4.20) and (4.22) are the desired expressions for the expansion coefficients $a_n(x)$ for cases where the modulating function $\sigma(x)$ is considered to be deterministic.

Notice from Eq. (4.22) that evaluation at location x of the instantaneous spectrum $\Phi_W(k,x)$ by the finite series expansion of Eq. (4.11) depends only on properties of $\sigma(x)$ measured at that same value of x. This property is not true of the most general representation of the instantaneous spectrum $\Phi_W(k,x)$, which does not depend on convergence of the series of Eq. (4.11). However, from a physical point of view, dependence of $\Phi_W(k,x)$ only on properties of $\sigma(\xi)$ in the immediate vicinity of $\xi = x$ is a very desirable property. Expressions for $a_0(x)$ to $a_8(x)$, obtained from Eq. 4.22, are written out in Appendix E.

Case of possibly nonhomogeneous stochastic process $\sigma(x)$. In cases where $\sigma(x)$ in Eq. (3.1a) is considered to be a possibly nonhomogeneous stochastic process that is independent of the homogeneous process z(x), the terms $a_n(x)$ may be obtained by taking the expected value of Eq. (4.22), i.e.,

$$a_n(x) = \frac{1}{(i4\pi)^n}\left\{2\left[\sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n}{k} \left\langle \frac{d^k\sigma(x)}{dx^k}\,\frac{d^{n-k}\sigma(x)}{dx^{n-k}} \right\rangle\right] + (-1)^{n/2}\binom{n}{n/2}\left\langle \left[\frac{d^{n/2}\sigma(x)}{dx^{n/2}}\right]^2 \right\rangle\right\} \quad , \quad n = \text{even} \quad , \quad (4.23)$$

and where, from Eq. (4.20), we have

$$a_n(x) = 0 \quad , \quad n = \text{odd} \quad . \qquad \text{(4.20 repeated)}$$

Case of ergodic process σ(x). According to Eq. (3.17a), in cases where $\sigma(x)$ is ergodic and independent of z(x), we may obtain the usual power spectral density $\Phi_w(k)$ of the process w(x) by taking the average value of $\Phi_w(k,x)$ with respect to x. From Eq. (4.11), it is evident that this averaging operation requires integrals of the coefficients $a_n(x)$. In Appendix F, it is shown [Eq. (F.14)] that for n = even, we have, upon introducing the definition (F.11),

$$\int_A^B a_n(x)dx = \frac{2}{(i4\pi)^n}\left[\sum_{\ell=0}^{\frac{n}{2}-1} (-1)^\ell\, F(\ell;n)\, \sigma^{(\ell)}(x)\, \sigma^{(n-1-\ell)}(x)\right]\Bigg|_{x=A}^{x=B} + \frac{1}{(2\pi)^n}\int_A^B [\sigma^{(n/2)}(x)]^2\, dx \quad , \quad n = \text{even} \quad , \quad (4.24)$$

where $\sigma^{(k)}(x)$ is the *kth* derivative of $\sigma(x)$. The first term in the right-hand side of Eq. (4.24) involves evaluation of the derivatives of $\sigma(x)$ through order n-1 evaluated at the endpoints of the interval $A \le x \le B$. For stationary processes whose derivatives through order n exist, the expected value of the first term will be independent of the length of the integration interval X = B-A, provided that X is larger than the correlation interval of the process $\sigma(x)$. Furthermore, since $[\sigma^{(n/2)}(x)]^2$ is necessarily positive, the second term in the right-hand side of Eq. (4.24) increases with increasing X = B-A, and for ergodic processes, this integral will become proportional to X = B-A as $X\to\infty$. Consequently, as $X\to\infty$, the first term will become negligible in comparison with the second term. Hence, in the case of ergodic processes, we have for the space average of $a_n(x)$ when the averaging interval is very large:

$$\langle a_n \rangle_x \triangleq \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} a_n(x)dx \tag{4.25a}$$

$$= \frac{1}{(2\pi)^n} \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} [\sigma^{(n/2)}(x)]^2 \, dx \tag{4.25b}$$

$$= \frac{1}{(2\pi)^n} \langle [\sigma^{(n/2)}(x)]^2 \rangle_x \quad , \quad n = \text{even} \quad , \tag{4.25c}$$

where $\langle \cdots \rangle_x$ denotes an average with respect to the variable x over an infinite interval as indicated by Eq. (4.25a). Noting that, in the case of ergodic processes, $\langle \phi_\sigma(\xi,x) \rangle_x = \phi_\sigma(\xi)$, as is indicated by Eq. (3.19), we have by combining Eqs. (3.19), (4.12), and (4.25c) and setting n = 2m,

$$\left. \frac{d^{2m}\phi_\sigma(\xi)}{d\xi^{2m}} \right|_{\xi=0} = (-1)^m \left\langle \frac{d^m\sigma(x)}{dx^m} \right\rangle_x^2 \quad , \quad m = 0,1,2,\cdots \quad , \tag{4.26}$$

which is a well known result; see, e.g., p. 21 of Bendat [*17*].

The result implied by the equality of Eqs. (4.25a) and (4.25b) does not directly depend on the ergodic hypothesis; rather, it applies to spectra in general that are computed from a single record by appropriate averaging over the coordinate x. In this interpretation, it is important to notice that if $\sigma(x)$ and its derivatives vanish at the endpoints of the averaging interval, the first term on the right-hand side of Eq. (4.24) vanishes, and the only result that contributes to the right-hand side of Eq. (4.24) is the integral of $[\sigma^{(n/2)}(x)]^2$ provided by the second term. Furthermore, in this case, or when the contributions provided by the endpoints are negligible, it is immediately evident from Eq. (4.24) that the integral of every $a_n(x)$, $n = 0,2,4,\cdots$ is nonnegative. For ergodic processes, it is immediately evident from Eq. (4.25c) that $\langle a_n \rangle_x$, $n = 0,2,4,\cdots$ is always nonnegative.

Interpretation of Series Expansion

We shall now interpret the series expansion of Eq. (4.11) in terms of typical length scales of the processes $\sigma(x)$ and $z(x)$. We consider first deterministic modulating functions $\sigma(x)$. Consider a family of modulating functions $\sigma(x)$, each of which is identical with all others in the family except for a scale factor on the "x-axis". Then, if we define a length scale, L_σ, associated with this family, all members of the family become identical when plotted as a function of x/L_σ. Let us define the dimensionless independent variable associated with this family of functions by

$$\bar{x} \triangleq x/L_\sigma \quad ; \tag{4.27}$$

hence, we have $x = L_\sigma \bar{x}$. Then, the family of functions of the dimensionless coordinate $\bar{x}$, defined by

$$\bar{\sigma}(\bar{x}) \triangleq \sigma(L_\sigma \bar{x}) \quad , \tag{4.28}$$

are all identical (by definition). This fact is, perhaps, more easily seen if we substitute Eq. (4.27) into (4.28); i.e.,

$$\sigma(x) = \bar{\sigma}(x/L_\sigma) \quad , \tag{4.29}$$

where we have reversed the two sides of the equation.

Consider, now, the expansion coefficient, $a_n(x)$, defined by Eq. (4.19), which requires the derivatives of $\sigma(x)$ for its evaluation. Define

$$\bar{\sigma}^{(k)}(\bar{x}) \triangleq \frac{d^k\bar{\sigma}(\bar{x})}{d\bar{x}^k} \quad . \tag{4.30}$$

Then, it immediately follows by repeated differentiation of Eq. (4.29), using the "chain rule", that

$$\frac{d^k\sigma(x)}{dx^k} = \left(\frac{1}{L_\sigma}\right)^k \bar{\sigma}^{(k)}(x/L_\sigma) \quad . \tag{4.31}$$

Consequently, using Eq. (4.31), Eq. (4.19) may be written as

$$a_n(x) = \left(\frac{1}{L_\sigma}\right)^n \frac{1}{(-i4\pi)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \bar{\sigma}^{(k)}(x/L_\sigma)\, \bar{\sigma}^{(n-k)}(x/L_\sigma) \quad , \tag{4.32}$$

or, changing to the dimensionless variable $\bar{x}$ on the right-hand side, Eq. (4.32) may be expressed as

$$a_n(x) = \left(\frac{1}{L_\sigma}\right)^n \frac{1}{(-i4\pi)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \bar{\sigma}^{(k)}(\bar{x})\, \bar{\sigma}^{(n-k)}(\bar{x})$$

$$\equiv \left(\frac{1}{L_\sigma}\right)^n \bar{a}_n(\bar{x}) \quad . \tag{4.33}$$

Now, since all members $\bar{\sigma}(\bar{x})$ of the family of functions are identical, it follows that the right-hand side of Eq. (4.33) is identical for all members of the family, except for the term $(1/L_\sigma)^n$. Consequently, it follows from Eq. (4.33) that for functions $\sigma(x)$, identical except for a scale factor L_σ on the abscissa, the coefficients $a_n(x)$, evaluated at "equivalent positions" $\bar{x}$, are proportional to $(1/L_\sigma)^n$. Integrals of $a_n(x)$ are proportional to $(1/L_\sigma)^{n-1}$. Notice that we may consider the (identical) functions $\bar{\sigma}(\bar{x})$ as all having the same length scale of unity in the variable $\bar{x}$.

For homogeneous stochastic processes $\sigma(x)$, let us define the autocorrelation function of $\sigma(x)$, as before, by

$$\phi_\sigma(\xi) \triangleq \langle\sigma(x)\ \sigma(x+\xi)\rangle \quad . \tag{4.34}$$

Let us now consider a family of *stochastic processes* $\sigma(x)$, where each member *process* of the family is identical to the others except for a scale factor L_σ on the abscissa. This implies that the normalized autocorrelation function, defined as

$$\bar{\phi}_\sigma(\bar{\xi}) \triangleq \langle\sigma(L_\sigma\ \bar{x})\ \sigma(L_\sigma\ \bar{x} + L_\sigma\ \bar{\xi})\rangle \tag{4.35a}$$

$$= \phi_\sigma(L_\sigma\ \bar{\xi}) \quad , \tag{4.35b}$$

is the same for all stochastic processes $\sigma(x)$ in the family, where we have defined the dimensionless lag variable $\bar{\xi}$ as

$$\bar{\xi} \triangleq \xi/L_\sigma \quad . \tag{4.36}$$

Equation (4.35b) may be expressed in terms of the variable ξ by

$$\phi_\sigma(\xi) = \bar{\phi}_\sigma(\xi/L_\sigma) \quad , \tag{4.37}$$

where we have again reversed the two sides of the equation. Defining

$$\overline{\phi}_\sigma^{(k)}(\overline{\xi}) \triangleq \frac{d^k \overline{\phi}_\sigma(\overline{\xi})}{d\overline{\xi}^k} \quad , \tag{4.38}$$

we have from Eqs. (4.36) and (4.37),

$$\frac{d^n \phi_\sigma(\xi)}{d\xi^n} = \left(\frac{1}{L_\sigma}\right)^n \overline{\phi}_\sigma^{(n)}(\xi/L_\sigma) \quad . \tag{4.39}$$

Applying Eq. (4.39) to Eq. (4.12) for the homogeneous case where $\phi_\sigma(\xi,x)$ is a function of ξ only, as indicated by Eq. (3.13), we have

$$a_n(x) = \left(\frac{1}{L_\sigma}\right)^n \frac{1}{(-i2\pi)^n} \overline{\phi}_\sigma^{(n)}(\overline{\xi}) \Big|_{\overline{\xi}=0} \quad . \tag{4.40}$$

Consequently, for homogeneous processes that are identical except for a scale factor L_σ on the independent variable x, the coefficients $a_n(x)$ are all identical with each other except for a multiplicative constant $(1/L_\sigma)^n$, as was the case for deterministic $\sigma(x)$. A comparable result can be proved from Eq. (4.12) for nonhomogeneous stochastic processes $\sigma(x)$.

Consider now the homogeneous process z(x) in Eq. (3.1). Again, we consider a family of processes z(x), where each member process is identical to the others except for a scale factor L_z on the abscissa. Let $\phi_z(\xi)$ be the autocorrelation function of one of these processes. Define a dimensionless independent variable associated with the process z(x) by

$$\overline{x} = \frac{x}{L_z} \quad , \tag{4.41}$$

and a dimensionless lag variable by

$$\overline{\xi} = \frac{\xi}{L_z} \quad . \tag{4.42}$$

Then, by arguments identical to those above for the stochastic process $\sigma(x)$, it follows that the normalized autocorrelation function $\overline{\phi}_z(\overline{\xi})$, defined as

$$\overline{\phi}_z(\overline{\xi}) \triangleq \phi_z(L_z\overline{\xi}) \quad , \tag{4.43}$$

is the same for every one of the stochastic processes $z(x)$ in the family. Expressing Eq. (4.43) in terms of the dimensional lag variable ξ gives

$$\phi_z(\xi) = \overline{\phi}_z(\xi/L_z) \quad , \tag{4.44}$$

where we have again reversed the two sides of Eq. (4.43) in going to Eq. (4.44).

Let us now form the Fourier transform of $\phi_z(\xi)$ [which is the wavenumber spectrum of the process $z(x)$], and then use Eq. (4.44):

$$\begin{aligned}
\Phi_z(k) &= \int_{-\infty}^{\infty} \phi_z(\xi) e^{-i2\pi k\xi} d\xi \\
&= L_z \int_{-\infty}^{\infty} \overline{\phi}_z(\xi/L_z) e^{-i2\pi k L_z(\xi/L_z)} d\xi/L_z \\
&= L_z \int_{-\infty}^{\infty} \overline{\phi}_z(\overline{\xi}) e^{-i2\pi L_z k\overline{\xi}} d\overline{\xi} \\
&= L_z \, \overline{\Phi}_z(L_z k) \quad ,
\end{aligned} \tag{4.45}$$

where we have defined

$$\overline{\Phi}_z(\overline{k}) \triangleq \int_{-\infty}^{\infty} \overline{\phi}_z(\overline{\xi}) e^{-i2\pi\overline{k}\overline{\xi}} d\overline{\xi} \quad , \tag{4.46}$$

where $\overline{k} \triangleq L_z k$ is a dimensionless wavenumber. According to Eqs. (4.9) and (4.11), we are interested in the derivatives of $\Phi_z(k)$. Defining

$$\overline{\Phi}_z^{(n)}(\overline{k}) \triangleq \frac{d^n\overline{\Phi}_z(\overline{k})}{d\overline{k}^n} \quad , \tag{4.47}$$

we have, upon using Eq. (4.45) and the "chain rule",

$$\Phi_z^{(n)}(k) = L_z^{n+1}\,\overline{\Phi}_z^{(n)}(\overline{k}) \tag{4.48a}$$

$$= L_z^{n+1}\,\overline{\Phi}_z^{(n)}(L_z k) \quad . \tag{4.48b}$$

Now, since $\overline{\phi}_z(\overline{\xi})$ is the same for all members of the family of stochastic processes $z(x)$ under consideration, it follows from Eq. (4.46) that $\overline{\Phi}_z(\overline{k})$ and, therefore, its derivatives are also the same for all members. Thus, it follows from Eq. (4.48a) that the *nth* derivative of the wavenumber spectrum of any one of the members of the family of processes is equal to L_z^{n+1} times a function that is common to all members of the family.

Let us now combine Eqs. (4.33) and (4.48b) with the series expansion, Eq. (4.11):

$$\Phi_w(k,x) = \sum_{n=0}^{N} \left(\frac{L_z}{L_\sigma}\right)^n \frac{\overline{a}_n(x/L_\sigma)}{n!}\, L_z\, \overline{\Phi}_z^{(n)}(L_z k) + R_{N+1}(k,x) \quad , \tag{4.49}$$

where $\overline{a}_n(\overline{x})$ and $\overline{\Phi}_z^{(n)}(\overline{k})$ are the same for all members of the families of processes that they respectively describe. *It follows that the rate of convergence of the series in Eqs. (4.11) and (4.49) is controlled by the ratio of length scales* L_z/L_σ, *and that when* $(L_z/L_\sigma) << 1$, *convergence should be rapid.*

Let us now examine the first term in Eq. (4.11). Evaluating Eq. (4.19) for $n = 0$, we see immediately that, for deterministic $\sigma(x)$, we have

$$a_0(x) = \sigma^2(x) \quad . \tag{4.50}$$

Hence, the first term in Eq. (4.11) is $\sigma^2(x)\,\Phi_z(k)$, which according to Eqs. (4.49), (4.33), and (4.45) is the limiting form of the instantaneous spectrum $\Phi_w(k,x)$ as $(L_z/L_\sigma) \to 0$; that is,

$$\Phi_w(k,x) \approx \sigma^2(x)\,\Phi_z(k) \quad , \qquad (L_z/L_\sigma) \to 0 \quad , \tag{4.51}$$

as we might expect on intuitive grounds. Equation (4.51) is the usual quasi-stationary (or more appropriately, quasi-homogeneous) spectrum approximation. The above arguments provide a rigorous justification of this approximation.

In order to determine when nonhomogeneous behavior begins to have an effect on the *form* of the instantaneous wavenumber spectrum, we need to include one correction term to the approximation, Eq. (4.51). According to Eq. (4.19), we have

$$a_2(x) = -\frac{1}{8\pi^2}\{\sigma(x)\,\sigma''(x) - [\sigma'(x)]^2\} \tag{4.52a}$$

$$= -\frac{1}{8\pi^2}\,\sigma^2(x)\,\frac{d^2\ell n\sigma(x)}{dx^2} \quad ; \tag{4.52b}$$

hence, including only one correction term to the quasi-homogeneous approximation, Eq. (4.51), we have from Eqs. (4.11), (4.51), and (4.52b),

$$\Phi_w(k,x) \approx \sigma^2(x)\left[\Phi_z(k) - \frac{1}{16\pi^2}\,\frac{d^2\ell n\sigma(x)}{dx^2}\,\Phi_z^{(2)}(k)\right] ,$$

$$(L_z/L_\sigma) << 1 \quad . \tag{4.53}$$

By comparing the magnitudes of the two terms within the brackets in Eq. (4.53), we can determine when nonhomogeneous behavior becomes sufficiently rapid to change the form of the wavenumber spectrum $\Phi_z(k)$.

It is evident from Eq. (4.12) that the second term within the brackets in Eq. (4.53) arises from the second derivative with respect to ξ of the instantaneous autocorrelation function $\phi_\sigma(\xi,x)$ evaluated at $\xi = 0$. Consequently, if $\phi_\sigma(\xi,x)$ is well approximated by a quadratic function of ξ over the range of ξ where $\phi_z(\xi)$ is not negligible, then the approximation of Eq. (4.53) should provide good results. Figure 6 illustrates such a situation.

Expansion Functions for von Karman Spectra

In order to evaluate the series expansion of Eq. (4.11), it is necessary to calculate the derivatives of the power spectrum $\Phi_z(k)$ of the homogeneous component $z(x)$. The case of most interest is the von Karman transverse spectrum which has the general form

$$\Phi_z(k) = L_z\,\overline{\Phi}_z(\overline{k})$$

$$= L_z\,\overline{\Phi}_z(L_z k) \quad , \tag{4.54}$$

$$\bar{k} = L_z k \quad , \tag{4.55}$$

where

$$\bar{\Phi}_z(\bar{k}) = \frac{A + B\,\bar{k}^2}{(1 + C\,\bar{k}^2)^{11/6}} \tag{4.56a}$$

$$= f_1(\bar{k})\; f_2(\bar{k}) \quad , \tag{4.56b}$$

where

$$f_1(\bar{k}) = A + B\,\bar{k}^2 \tag{4.57}$$

$$f_2(\bar{k}) = (1 + C\,\bar{k}^2)^{-11/6} \quad , \tag{4.58}$$

and where, for a two-sided spectrum satisfying [see Eq. (3.1b)]

$$\int_{-\infty}^{\infty} \Phi_z(k)dk = \langle z^2 \rangle = 1 \quad , \tag{4.59}$$

we have*

$$A = 1$$

$$B = 188.75$$

$$C = 70.78 \quad . \tag{4.60}$$

The quantity L_z is the integral scale of the turbulence.

Using Leibniz's rule for the *nth* derivative of a product, we may express the *nth* derivative of $\bar{\Phi}_z(\bar{k})$ of Eq. (4.56b) as

$$\bar{\Phi}_z^{(n)}(\bar{k}) = \sum_{j=0}^{n} \binom{n}{j}\, f_1^{(j)}(\bar{k})\; f_2^{(n-j)}(\bar{k}) \quad , \tag{4.61}$$

*Exact values of the constants B and C in Eq. (4.60) are given by the left-hand sides of Eqs. (C.4) and (C.3), respectively.

where

$$f^{(j)}(\bar{k}) \triangleq \frac{d^j f(\bar{k})}{d\bar{k}^j} \quad . \tag{4.62}$$

Applying Eq. (4.61) to Eq. (4.56b) for n = 0,2,4,6, and 8, we obtain

$$\begin{aligned}
\bar{\Phi}_z^{(0)}(\bar{k}) &= (A + B\,\bar{k}^2)\; f_2(\bar{k}) \\
\bar{\Phi}_z^{(2)}(\bar{k}) &= (A + B\,\bar{k}^2)\; f_2^{(2)}(\bar{k}) + 4B\,\bar{k}\; f_2^{(1)}(\bar{k}) + 2B\; f_2(\bar{k}) \\
\bar{\Phi}_z^{(4)}(\bar{k}) &= (A + B\,\bar{k}^2)\; f_2^{(4)}(\bar{k}) + 8B\,\bar{k}\; f_2^{(3)}(\bar{k}) + 12B\; f_2^{(2)}(\bar{k}) \\
\bar{\Phi}_z^{(6)}(\bar{k}) &= (A + B\,\bar{k}^2)\; f_2^{(6)}(\bar{k}) + 12B\,\bar{k}\; f_2^{(5)}(\bar{k}) + 30B\; f_2^{(4)}(\bar{k}) \\
\bar{\Phi}_z^{(8)}(\bar{k}) &= (A + B\,\bar{k}^2)\; f_2^{(8)}(\bar{k}) + 16B\,\bar{k}\; f_2^{(7)}(\bar{k}) + 56B\; f_2^{(6)}(\bar{k}) \quad ,
\end{aligned} \tag{4.63}$$

where the first eight derivatives of $f_2(\bar{k})$ are listed in Appendix G.

The expansion functions $\Phi_z^{(n)}(k)$ in Eq. (4.11) can be computed from the above expressions using Eq. (4.48b). Thus, except for the amplitude and wavenumber scales, the functions $\bar{\Phi}_z^{(n)}(\bar{k})$ have the same form as the functions $\Phi_z^{(n)}(k)$. The functions $\bar{\Phi}_z^{(n)}(\bar{k})/|\bar{\Phi}_z^{(n)}(0)|$, computed using the above formulas, are plotted in Fig. 7.

The functions plotted in Fig. 7 are related to the actual expansion functions by

$$\frac{\Phi_z^{(n)}(k)}{|\Phi_z^{(n)}(0)|} = \frac{\bar{\Phi}_z^{(n)}(L_z k)}{|\bar{\Phi}_z^{(n)}(0)|} \quad , \tag{4.64}$$

as may be seen from Eq. (4.48b). Consequently, the *signs* of the functions plotted in Fig. 7 are the same as the signs of the functions $\Phi_z^{(n)}(\bar{k}/L_z)$. This fact is important for the following reason. It was pointed out earlier, under quite general conditions, that integrals of the expansion coefficients $a_n(x)$ are necessarily non-negative. Consequently, in using the series expansion of Eq. (4.11) to predict the wavenumber spectra of experimental records,

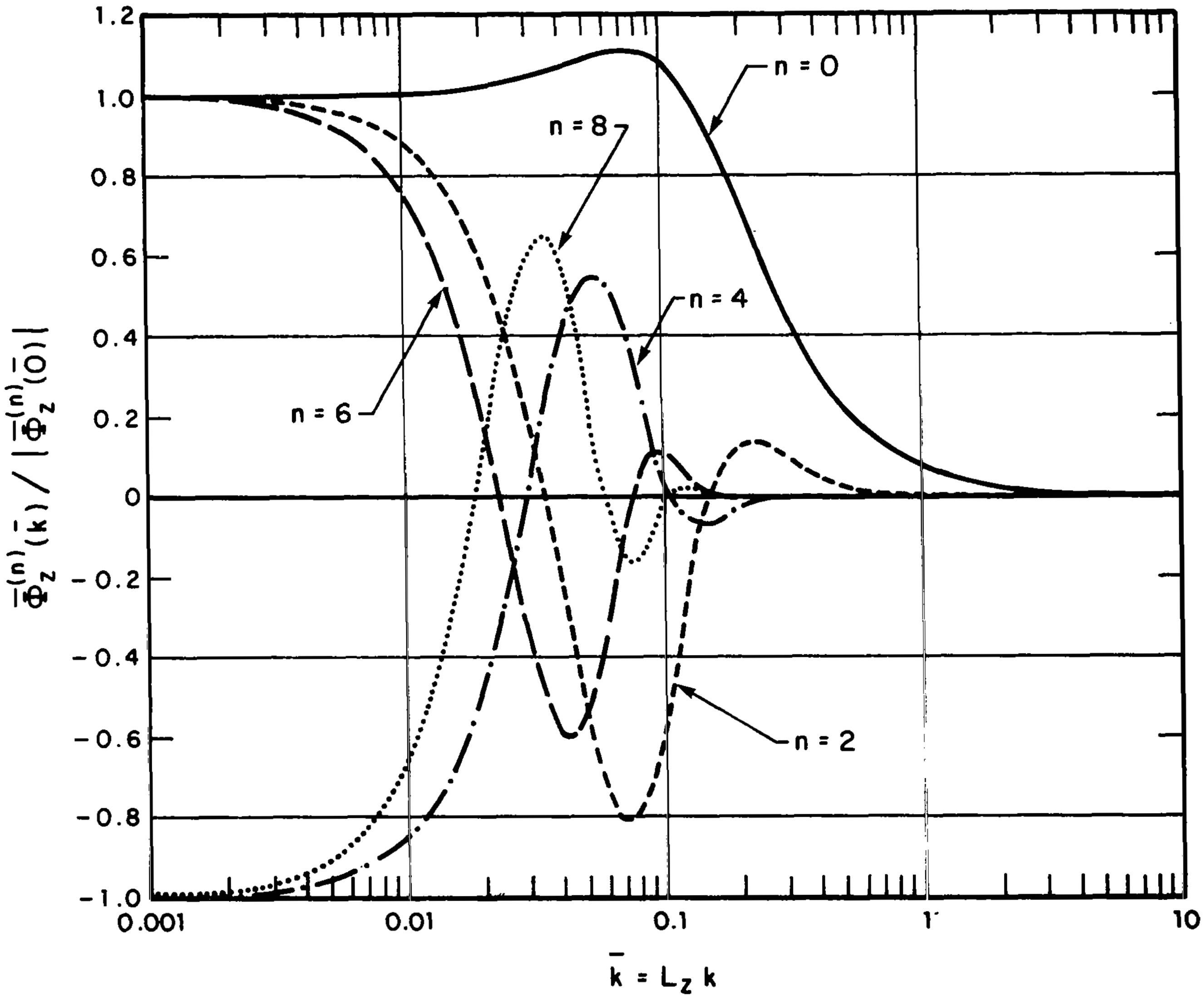

FIG. 7. NORMALIZED EXPANSION FUNCTIONS $\overline{\Phi}_z^{(n)}(\overline{k})/|\overline{\Phi}_z^{(n)}(\overline{0})|$ FOR THE von KARMAN TRANSVERSE SPECTRUM.

the corrections to the von Karman spectra provided by the terms $n = 2,4,6,\cdots$ in Eq. (4.11) have the same *signs* as the signs of the expansion functions plotted in Fig. 7. The spectrum shown in the semi-log plot of Fig. 7 for $n = 0$ is the von Karman transverse spectrum. The first correction term in Eq. (4.11) to the von Karman spectrum is proportional to the curve in Fig. 7 for $n = 2$. Thus, when a fraction of the curve for $n = 2$ is added to the curve for $n = 0$, the resulting spectrum is larger than the curve for $n = 0$ for very small values of k, but is smaller than the curve for $n = 0$ in the neighborhood of the knee. Thus, under quite general conditions, we have shown that for mild nonhomogeneous effects predicted by the two term approximation of Eq. (4.53), the second term must have the effect of smoothing the knee of the von Karman spectrum. This general effect of nonhomogeneous (i.e., nonstationary) behavior is well known.

Finally, we note that the values of $\overline{\Phi}_z^{(n)}(0)$ are

$$\overline{\Phi}_z(0) = 1.0$$

$$\overline{\Phi}_z^{(2)}(0) = 117.97$$

$$\overline{\Phi}_z^{(4)}(0) = -2.755 \times 10^5$$

$$\overline{\Phi}_z^{(6)}(0) = 9.210 \times 10^8$$

$$\overline{\Phi}_z^{(8)}(0) = -4.898 \times 10^{12} \quad . \tag{4.65}$$

Examples: Abrupt Onset of Turbulence and Burst of Turbulence

Abrupt onset of turbulence. As a first example of nonhomogeneous turbulence, we consider the case where the modulating function $\sigma(x)$ of Eq. (3.1) rises abruptly from zero to unity. A convenient mathematical representation of such behavior is

$$\sigma(x) = \frac{1}{2}\left[1 + \tanh\left(2x/L_\sigma\right)\right] \quad . \tag{4.66}$$

which is illustrated in Fig. 8. Notice from Fig. 8 that L_σ is the nominal distance associated with the rise: L_σ is defined as the distance required for a straight line approximation to $\sigma(x)$ to rise from zero to unity with slope equal to the slope of $\sigma(x)$ evaluated at the mid-rise position $x = 0$. Notice that $\sigma(x)$ remains at unity for arbitrarly large values of x.

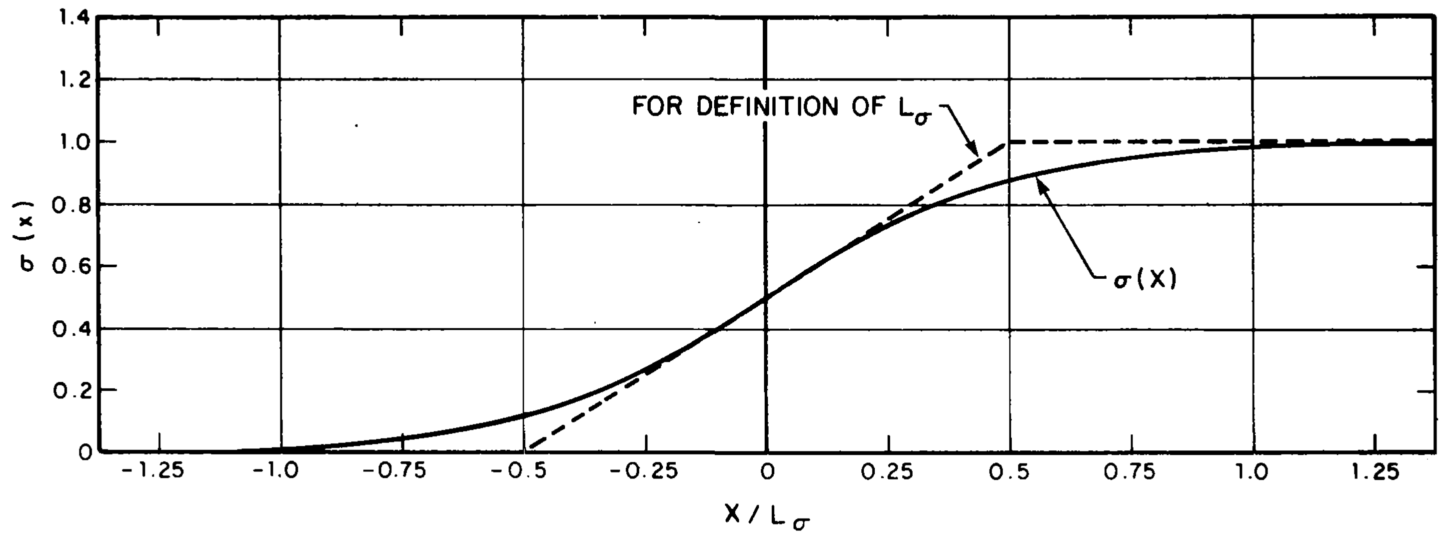

FIG. 8. MODULATING FUNCTION REPRESENTING ABRUPT ONSET OF TURBULENCE.

To evaluate the effect of this nonhomogeneous behavior of $\sigma(x)$ on the instantaneous spectrum $\Phi_W(k,x)$, we need to evaluate $a_2(x)$ given by Eq. (4.52a). It is easy to show from Eq. (4.66) that the first two derivatives of $\sigma(x)$ are

$$\sigma'(x) = (1/L_\sigma)\ \mathrm{sech}^2\ (2x/L_\sigma) \tag{4.67a}$$

and

$$\sigma''(x) = -(4/L_\sigma^2)\ \mathrm{sech}^2\ (2x/L_\sigma)\ \tanh\ (2x/L_\sigma) \quad . \tag{4.67b}$$

Using Eqs. (4.66) and (4.67), it is shown in Appendix H that

$$a_2(x) = \frac{\sigma^2(x)}{2\pi^2 L_\sigma^2}\ \mathrm{sech}^2\ (2x/L_\sigma) \quad . \tag{4.68}$$

Therefore, using Eqs. (4.50) and (4.68), we may express the two-term approximation to $\Phi_W(k,x)$ as

$$\Phi_W(k,x) \approx a_0(x)\ \Phi_Z(k) + \frac{1}{2}\ a_2(x)\ \Phi_Z^{(2)}(k) \tag{4.69a}$$

$$= \sigma^2(x) \left[\Phi_Z(k) + \frac{\mathrm{sech}^2\ (2x/L_\sigma)}{4\pi^2 L_\sigma^2}\ \Phi_Z^{(2)}(k)\right] \tag{4.69b}$$

$$= \sigma^2(x)\ L_Z \left[\overline{\Phi}_Z(L_Z k) + \frac{1}{4\pi^2}\left(\frac{L_Z}{L_\sigma}\right)^2 \mathrm{sech}^2\ (2x/L_\sigma)\ \overline{\Phi}_Z^{(2)}(L_Z k)\right] \quad , \tag{4.69c}$$

where we have used Eq. (4.48b) in going to the last step.

The first term $\sigma^2(x)\ L_Z\ \overline{\Phi}_Z(L_Z k)$ in the right-hand side of Eq. (4.69c) is the quasi-homogeneous approximation to the instantaneous spectrum $\Phi_W(k,x)$, valid when $(L_Z/L_\sigma) \to 0$. Notice that the coefficient $a_2(x)$, given by Eq. (4.68), is nonnegative* for all

*It was shown in Sec. 4.2 that infinite *integrals* of the $a_n(x)$ must be positive; however, for some values of x, the coefficients $a_n(x)$ can, in general, be negative.

values of x, and that the coefficient to the correction term in Eq. (4.69c) is proportional to the ratio of length scales $(L_z/L_\sigma)^2$, as was shown, for general processes, by Eq. (4.49).

The dependence on x of the correction term in Eq. (4.69c) is proportional to $\sec^2(2x/L_\sigma)$ which is plotted in Fig. 9. From Fig. 9, it is evident that the correction term has appreciable influence only in the region $-(L_\sigma/2) < x < (L_\sigma/2)$, where $\sigma(x)$ itself displays nonhomogeneous behavior, as may be seen from Fig. 8.

Since the sign of $a_2(x)$, Eq. (4.68), is everywhere nonnegative, it is easily seen from Eq. (4.69a) and Fig. 7 that the nonhomogeneous behavior of the present example has the effect of smoothing the knee the von Karman spectrum for all values of x. Noting from Eqs. (4.65) that $\Phi_z(0) = 1$ and $\Phi_z^{(2)}(0) = 117.97$ and from Fig. 9 that $\mathrm{sech}^2(0) = 1$, it is evident from Fig. 7 and Eq. (4.69c) that the ratio of the maximum contribution of the correction term to the quasi-homogeneous term occurs at k = 0, and is

$$C = \frac{117.97}{4\pi^2}\left(\frac{L_z}{L_\sigma}\right)^2$$

$$\approx 3.0\left(\frac{L_z}{L_\sigma}\right)^2 \quad . \qquad (4.70)$$

It may be seen from the curves for n = 0 and n = 2 in Fig. 7 that when $C \geq 0.10$, the correction term in Eq. (4.69c) will exhibit appreciable smoothing of the knee. Solving Eq. (4.70) for (L_z/L_σ) when C = 0.10, we find $(L_z/L_\sigma) \approx 0.2$ for this case. Consequently, *when* $L_\sigma \leq 5\ L_z$, *the instantaneous spectrum of the nonhomogeneous turbulence with modulating function* $\sigma(x)$ *shown in Fig. 8 will show a strongly rounded knee at* x = 0. On the other hand, when $C \leq 0.03$, the smoothing of the knee caused by the correction term will be barely discernible. Solving Eq. (4.70) for (L_z/L_σ) for this case, we find that *when* $L_\sigma \geq 10\ L_z$, *the nonhomogeneous effects will be virtually undetectable in the instantaneous spectrum for the modulating function* $\sigma(x)$ *shown in Fig. 8.*

Burst of turbulence. For a second example, we consider the case where the modulating function $\sigma(x)$ of Eq. (3.1) rises abruptly from zero to unity, and then falls abruptly back to zero. A convenient model for this behavior is the Gaussian function.

$$\sigma(x) = e^{-\pi(x/L_\sigma)^2} \quad , \qquad (4.71)$$

which is illustrated in Fig. 10. The coefficient π in the exponent of Eq. (4.71) was chosen so that the running integral $\int_{-\infty}^{x}\sigma(\xi)d\xi$ has

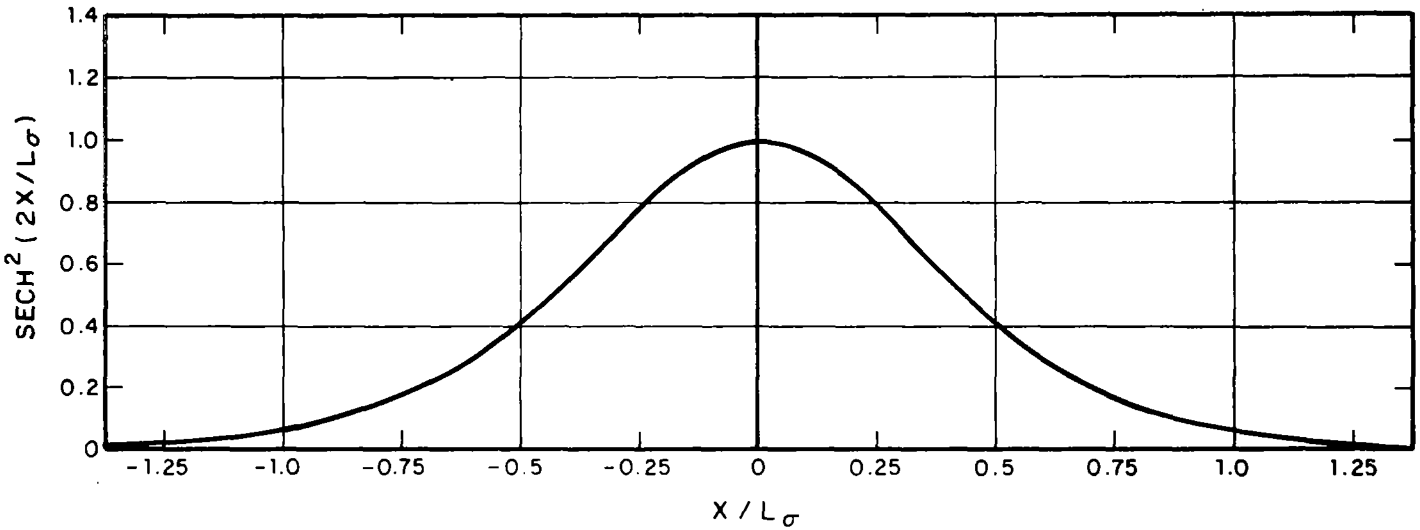

FIG. 9. THE CORRECTION TO THE QUASI-HOMOGENEOUS APPROXIMATION IN EQ. (4.69) IS PROPORTIONAL TO $\mathrm{sech}^2\ (2x/L_\sigma)$.

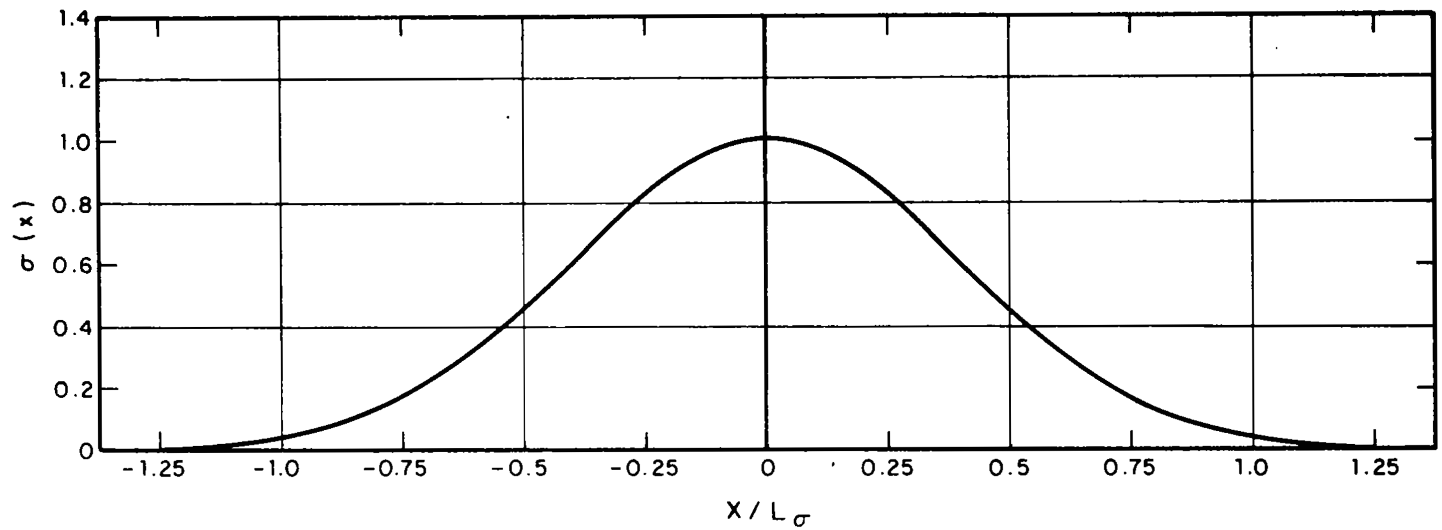

FIG. 10. MODULATING FUNCTION REPRESENTING BURST OF TURBULENCE.

the same scale as the function shown in Fig. 8; that is, the tangent to $\int_{-\infty}^{x} \sigma(\xi)d\xi$ at $x = 0$ grows from zero to unity within a span of exactly L_σ units of x.

Using Eq. (4.52b) to evaluate $a_2(x)$ in this case, we find immediately from Eq. (4.71) that $d^2 \ell n\sigma(x)/dx^2 = -2\pi/L_\sigma^2$; hence, we have

$$a_2(x) = \frac{\sigma^2(x)}{4\pi \; L_\sigma^2} \quad , \tag{4.72}$$

which is everywhere nonnegative, as was the case in the previous example. From Eq. (4.69a), we therefore have for our two-term approximation to $\Phi_w(k,x)$ in this case

$$\Phi_w(k,x) \approx \sigma^2(x) \left[\Phi_z(k) + \frac{1}{8\pi \; L_\sigma^2} \; \Phi_z^{(2)}(k) \right] \tag{4.73a}$$

$$= \sigma^2(x) \; L_z \left[\overline{\Phi}_z(L_z k) + \frac{1}{8\pi} \left(\frac{L_z}{L_\sigma} \right)^2 \overline{\Phi}_z^{(2)}(L_z k) \right] \quad , \tag{4.73b}$$

where we have again used Eq. (4.48) in going to the second step. Equation (4.73b) is of the general form of Eq. (4.49), as before. The relative weight of the correction term to the quasi-homogeneous approximation, in this case, is independent of x.

Again noting from Eqs. (4.65) that $\overline{\Phi}_z(0) = 1$ and $\overline{\Phi}_z^{(2)}(0) = 117.97$, it is evident from Fig. 7 and Eq. (4.73b) that the ratio of the contribition of the correction term in Eq. (4.73b) to the quasi-homogeneous term evaluated at $k = 0$ is

$$C = \frac{117.97}{8\pi} \left(\frac{L_z}{L_\sigma} \right)^2$$

$$\approx 4.7 \left(\frac{L_z}{L_\sigma} \right)^2 \quad . \tag{4.74}$$

Using the same reasoning as in the previous example, *we find for the present example that when* $L_\sigma \leq 7 \; L_z$, *the instantaneous spectrum will show a strongly rounded knee (for all values of x), whereas when* $L_\sigma \geq 13 \; L_z$, *the nonhomogeneous effects will be virtually undetectable in the spectrum.*

In the first example of the modulating function illustrated in Fig. 8, in integration of the instantaneous spectrum $\Phi_w(k,x)$ with respect to x to obtain the counterpart to the usual spectrum computed from experimental measurements, the effects of the nonhomogeneous rise illustrated in Fig. 8 would soon be averaged out in the integration, so that the effects of even a very rapid rise might not show up noticeably in a spectrum obtained from measured turbulence velocities.* However, the modulating function illustrated in Fig. 10 does not approach an asymptotically constant value (other than zero); consequently, any effects on the spectrum predicted by Eq. (4.73b) would show up in an integrated instantaneous spectrum as well; that is, in the ordinary spectrum computed from measured velocities.

Application to a Recorded Velocity History

It is of considerable interest to determine if measured records of atmospheric turbulence have nonstationary effects occurring sufficiently rapidly for such effects to be manifested in their spectra. For most records, it would seem that such nonstationary effects occur too slowly. However, the record shown in Fig. 11 has nonstationary behavior sufficiently rapid to show up in its spectrum.

Six "bursts" of turbulence are indicated by arrows in the vertical trace shown in Fig. 11. In each of these six bursts, the peak values appear to be at least four times the rms levels in the immediate neighborhoods of the bursts, and we must therefore regard their behavior as nonhomogeneous (or nonGaussian). The nominal *"durations"* τ_σ of these bursts are related to corresponding lengths L_σ by

$$\tau_\sigma = \frac{L_\sigma}{V} \quad , \tag{4.75}$$

where V is the measuring aircraft speed, which was 172.2 m/sec (565 ft/sec) for the record shown in Fig. 11. Consequently, applying the results of the previous example, we find that if

$$\tau_\sigma \leq 7 \frac{L_z}{V} \quad , \tag{4.76}$$

strong smoothing of the knee of the von Karman spectrum will be caused by the bursts, whereas, if

*This conclusion would appear to differ from the conclusion drawn by Houbolt [*1*]. See the discussion accompanying Houbolt's Fig. 19.

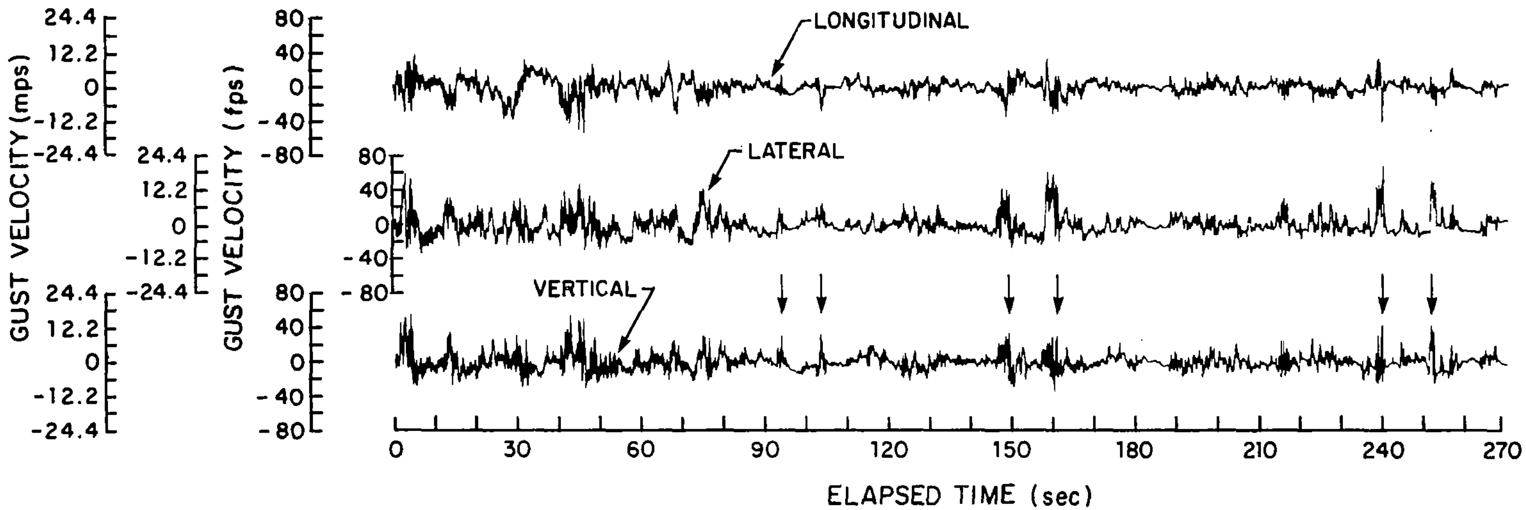

FIG. 11. TURBULENCE RECORDS FROM TEST NO. 168, LEG NO. 5 OF THE LO-LOCAT PROGRAM. (From Jones *et al*, p. 219)

$$\tau_\sigma \geq 13 \frac{L_z}{V} \quad , \tag{4.77}$$

the effects on the spectrum by the nonhomogeneous behavior will be difficult to detect.

The scale of the homogeneous component L_z of the vertical record shown in Fig. 11 was not computed. Let us assume that it is 137.16 m (450 ft), which is the scale of the vertical record shown in Fig. 2 and which is the scale determined from another low-altitude vertical record to be discussed in the next section. Thus, our criterion for strong smoothing of the knee is, from Eq. (4.76),

$$\tau_\sigma \leq \frac{7 \times 450}{565} = 5.6 \text{ sec} \quad , \tag{4.78}$$

whereas our criterion for negligible smoothing of the knee is

$$\tau_\sigma \geq \frac{13 \times 450}{565} = 10.4 \text{ sec} \quad . \tag{4.79}$$

Comparing the definition of τ_σ implied by Fig. 10 with the vertical record shown in Fig. 11, it is evident that the first two and last two bursts will cause strong smoothing of the knee of the von Karman spectrum since their nominal durations are of the order of $\tau_\sigma \approx 2$ sec, whereas the middle two bursts would probably cause weak but detectable smoothing, since their nominal durations are of the order of $\tau_\sigma \approx 7$ sec.

General Criterion for Negligible Effect of Nonhomogeneous Behavior on Shape of Spectrum

When the "correction term" expressed by the second term in the right-hand side of Eq. (4.53) is negligible in comparison with the first term for all values of x, the nonhomogeneous behavior manifested by variations in $\sigma(x)$ will not show up in the spectrum of $w(x)$ computed in the usual way. Combining Eqs. (4.48) and (4.53), we have

$$\Phi_w(k,x) \approx \sigma^2(x)\, L_z \left[\overline{\Phi}_z(L_z k) - \frac{L_z^2}{16\pi^2} \frac{d^2 \ell n\sigma(x)}{dx^2} \overline{\Phi}_z^{(2)}(L_z k) \right] \quad ; \tag{4.80}$$

consequently, our criterion for negligible effects of nonhomogeneous behavior on the shape of the spectrum is

$$\frac{L_z^2}{16\pi^2}\left|\frac{d^2 \ell n\sigma(x)}{dx^2}\right| |\overline{\Phi}_z^{(2)}(L_z k)| << \overline{\Phi}_z(L_z k) \quad . \tag{4.81}$$

Examination of the curves for n = 0 and n = 2 in Fig. 7 shows that the largest fractional contribution of $\overline{\Phi}_z^{(2)}(L_z k)$ occurs at the origin k = 0 and that if the contribution of the term corresponding to $\overline{\Phi}_z^{(2)}(0)$ is less than about 3% of the contribution $\overline{\Phi}_z^{(0)}(0)$, the correction term will cause negligible smoothing of the knee of the von Karman transverse spectrum. By combining Eqs. (4.65) and (4.81) with this fact, our criterion for negligible effect of the nonhomogeneous behavior is

$$\frac{L_z^2}{16\pi^2}\left|\frac{d^2 \ell n\sigma(x)}{dx^2}\right| (117.97) \leq 0.03$$

or

$$L_z^2 \left|\frac{d^2 \ell n\sigma(x)}{dx^2}\right| \leq 0.04 \quad , \tag{4.82}$$

where L_z is the integral scale of the homogeneous component z(x) in Eq. (3.1), which was assumed to have a von Karman transverse spectrum in arriving at the inequality in Eq. (4.82).

APPLICATION OF THE METHOD TO NONHOMOGENEOUS TURBULENCE RECORDS

In this section, the methods described in the previous two sections are applied to two nonhomogeneous turbulence recordings. The results are found to be consistent with the general conclusions drawn from the idealized examples described in Sec. 4.5.

Evaluation of Spectrum of Homogeneous Component of a "Slow" Burst of Turbulence Using the Arcsin Law

To illustrate the series expansion derived in the previous section and to verify the general conclusions drawn in Secs. 4.5 to 4.7, we use Eq. (4.11) to develop here the instantaneous spectrum of the "slow" burst of turbulence shown in the vertical record of Fig. 12.

The first step is to determine the spectrum of the homogeneous component z(t) of Eq. (3.1) using "infinite" clipping and the sine transformation of Eq. (3.20). The actual record used was the portion of the vertical record between the two vertical marks shown on the lowest trace in Fig. 12. The time interval between the vertical marks is from 62.00 to 110.65 sec, which represents a spatial interval of 9,144 m (30,000 ft), since the measured aircraft speed was 187.97 m/sec (616.7 ft/sec).

The procedure used to compute the spectrum of the stationary component was exactly the same as that described in Secs. 3.3, 3.4, and Appendix B of this report using the "infinite" clipping procedure. The value of M used in Eq. (B.1) was 962.4 m (3157.5 ft), which is equivalent to a duration of 5.12 sec or 512 discrete sample points of the record.

The spectrum computed from the infinitely clipped sample, corrected for clipping using the sine transformation of Eq. (3.20) and smoothed by the Papoulis window function described in Appendix B, is shown in Fig. 13. The spectrum shown in Fig. 13 is normalized so that it represents the wavenumber spectrum of a record with unit mean square value. (The mean value of the portion of the record between the vertical marks was computed and subtracted out before the clipping operation.)

To determine the integral scale of the homogeneous component of the vertical record shown in Fig. 12, the transverse von Karman spectrum was smoothed by the same Papoulis window used in the computation of the spectrum shown in Fig. 13. This procedure is described in Appendix C. The value of M used was 962.4 m (3157.5 ft), and the value of $\sigma^2 = 1$ was used, as was the case for the spectrum

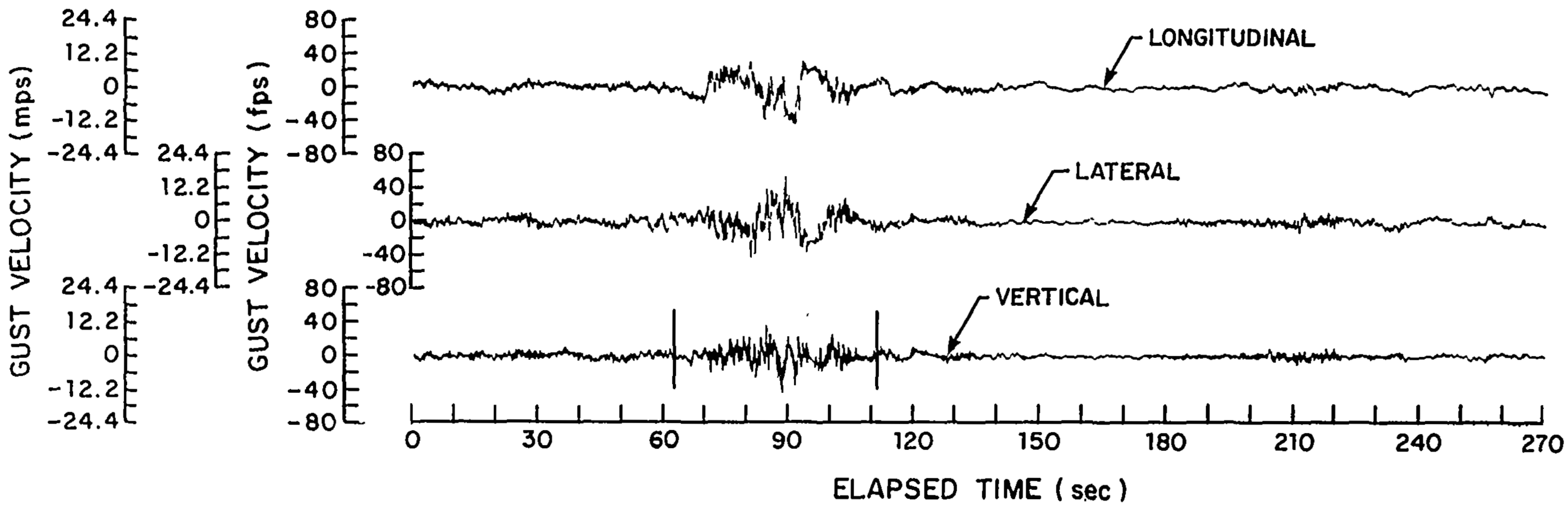

FIG. 12. NONHOMOGENEOUS TURBULENCE RECORDS FROM TEST NO. 191, LEG NO. 2 OF THE LO-LOCAT PROGRAM. (From Jones *et al*, p. 205)

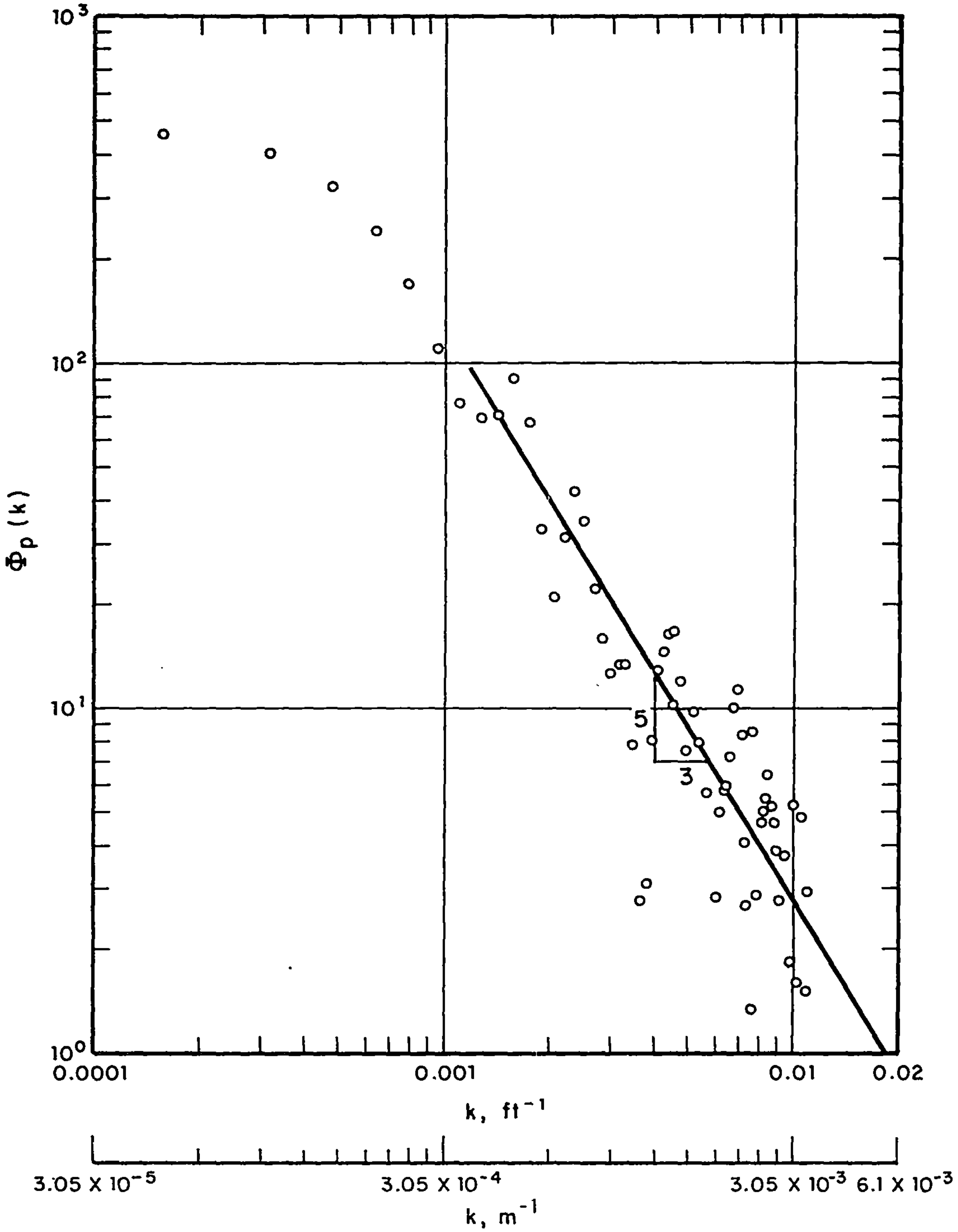

FIG. 13. WAVENUMBER SPECTRUM OF HOMOGENEOUS COMPONENT $z(x)$ OF THE PORTION BETWEEN THE TWO MARKS OF THE VERTICAL RECORD SHOWN IN FIG. 12. SPECTRUM NORMALIZED TO UNIT VARIANCE $\sigma^2 = 1$, AND SMOOTHED BY PAPOULIS WINDOW FUNCTION. SPECTRUM COMPUTED FROM CLIPPED RECORD.

shown in Fig. 13. The smoothed von Karman spectra are shown in Fig. 14 for integral scales of L = 121.92, 152.4, 182.88, 243.84, and 304.8 m (400, 500, 600, 800, and 1000 ft). The spectrum of Fig. 13 is also replotted in Fig. 14. It is evident that the spectrum of Fig. 13 conforms reasonably well to the von Karman spectrum with an integral scale of about 137.16 m (450 ft).

Method of Evaluation of Modulating Function $\sigma(x)$

The vertical record shown in Fig. 12 has a relatively smooth "envelope"; that is, the nonhomogeneous variance of the record monotonically increases to a maximum value in a smooth fashion and then monotonically decreases, also in a smooth fashion. This behavior suggests that it should be possible to represent adequately the nonhomogeneous standard deviation $\sigma(x)$ of the model of Eq. (3.1) by a polynomial of *4th* degree [i.e., represent adequately $\sigma(x)$ by a polynomial of *4th* degree* for the portion of the vertical record of Fig. 12 between the two marks].

The advantage of using a polynomial of finite degree to represent $\sigma(x)$ is that, for this representation, the series expansion, Eq. (4.11) contains a *finite* number of terms; therefore, there can be no problem with convergence of the expansion. If $\sigma(x)$ is a polynomial of degree N, then it is immediately evident from Eq. (4.22) that all $a_n(x)$ are identically zero for $n > 2N$, since every term in $a_n(x)$ contains a derivative of $\sigma(x)$ of order n/2 or higher. Consequently, when $\sigma(x)$ is a polynomial of *4th* degree, the series expansion of Eq. (4.11) reduces to

$$\Phi_w(k,x) = \sum_{\substack{n=0 \\ n=\text{even}}}^{8} \frac{a_n(x)}{n!} \Phi_z^{(n)}(k) \quad . \tag{5.1}$$

Expressions for $a_n(x)$, for n = 0 to 8, are contained in Appendix E. In using the expressions for the $a_n(x)$ in Appendix E, notice that the derivatives of $\sigma(x)$ of order larger than 4 are zero when $\sigma(x)$ is a polynomial of *4th* degree.

To estimate $\sigma(x)$ for the vertical record illustrated in Fig. 12, Legendre polynomials were used. First, the record was squared and the first four Legendre expansion coefficients, b'_m, of the "instantaneous" squared record w(x) were computed†:

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*A polynomial of *4th* degree is the lowest order polynomial that permits three stationary points, e.g., one maximum and two minima.

†In Eq. (5.2), the origin of the coordinate x has been chosen in the center of the expansion interval, i.e., x ranges from -D/2 to +D/2.

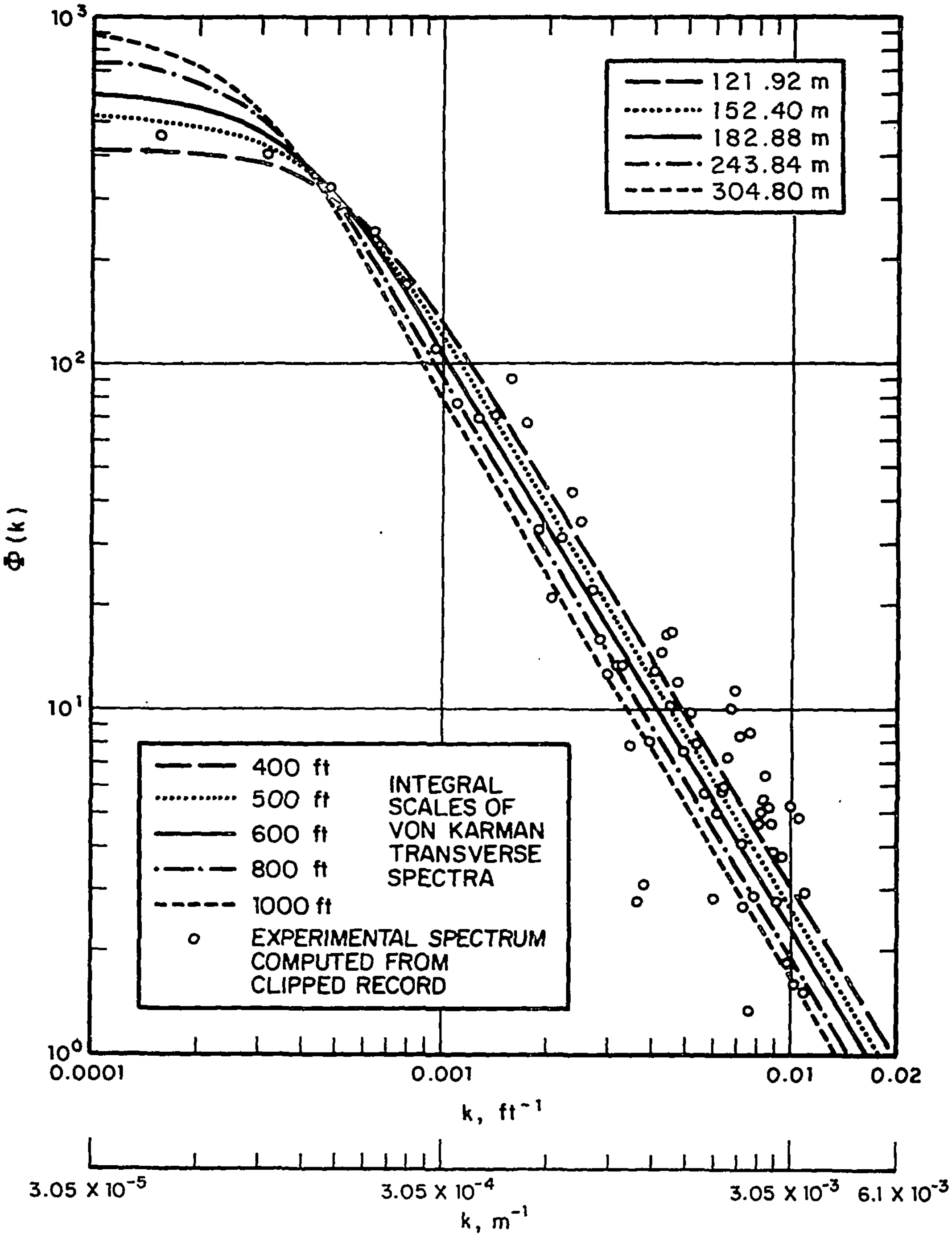

FIG. 14. COMPARISON OF SPECTRUM OF FIG. 13 WITH SMOOTHED von KARMAN SPECTRA POSSESSING SAME NORMALIZED VALUE OF $\sigma^2 = 1$. ALL SPECTRA WERE SMOOTHED BY SAME PAPOULIS WINDOW FUNCTION.

$$b'_m = \frac{2m+1}{D} \int_{-D/2}^{D/2} w^2(x)\, P_m(2x/D)\,dx \quad , \quad m = 0,1,2,3,4 \tag{5.2}$$

where the Legendre polynomials $P_m(\xi)$ used in Eq. 5.2 are

$$P_0(\xi) = 1$$

$$P_1(\xi) = \xi$$

$$P_2(\xi) = \frac{1}{2}(3\xi^2 - 1)$$

$$P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi)$$

$$P_4(\xi) = \frac{1}{8}(35\xi^4 - 30\xi^2 + 3) \quad . \tag{5.3}$$

The expansion interval, defined by the two lines on the vertical record shown in Fig. 12, is $(-D/2) < x < (D/2)$, where D = 9144 m (30,000 ft). The estimate $\hat{\sigma}^2(x)$ of the nonhomogeneous variance $\sigma^2(x)$ of the vertical record provided by the Legendre expansion is

$$\hat{\sigma}^2(x) = \sum_{m=0}^{4} b'_m\, P_m(2x/D) \tag{5.4}$$

The least squares property of Legendre polynomial expansions automatically insures that $\hat{\sigma}^2(x)$ is the least squares best fit of a *4th* degree polynomial to $w^2(x)$ within the 9144-m (30,000-ft) interval D.

To avoid convergence problems in the series expansion of $\Phi_W(k,x)$ it is necessary that $\sigma(x)$ [and not $\sigma^2(x)$] be a polynomial. To provide a polynomial representation of $\sigma(x)$, the square root of the *4th* degree polynomial of Eq. (5.4) was computed and then expanded in Legendre polynomials, whose expansion coefficients are, therefore,

$$b_m = \frac{2m+1}{D} \int_{-D/2}^{D/2} \sqrt{\hat{\sigma}^2(x)}\, P_m(2x/D)\,dx \quad , \quad m = 0,1,2,3,4 \quad . \tag{5.5}$$

Thus, the *4th* degree polynomial approximation to $\sigma(x)$ used in the remainder of the calculation is

$$\tilde{\sigma}(x) = \sum_{m=0}^{4} b_m \, P_m(2x/D) \quad , \tag{5.6}$$

where the polynomials $P_m(\xi)$ are defined by Eq. (5.3).

We may check the adequacy of the representation of Eq. (5.6) visually by plotting this representation on the original record. However, as is evident from the plot of the normal probability density function shown in Fig. 15, the value of 2σ is a better measure than σ of the *envelope* of a Gaussian random process. Consequently, we have plotted in Fig. 16 $2\tilde{\sigma}(x)$, computed by Eq. (5.6). The "random function" shown in Fig. 16 is an enlarged version of the 9144-m (30,000-ft) portion between the two marks of the vertical record shown in Fig. 12. Comparison of the curve $2\tilde{\sigma}(x)$ in Fig. 16 with the velocity record shown, and with the normal probability density of Fig. 15 would seem to indicate that $\tilde{\sigma}(x)$ is an excellent representation of the "instantaneous" standard deviation of the nonhomogeneous record.

Series Expansion of Instantaneous Spectra for Records of Finite Length

If we assume that the *4th* degree polynomial representation of $\sigma(x)$ provided by Eq. (5.6) is an exact description, then the representation of the instantaneous spectrum provided by Eq. (5.1) is also exact. However, the representation of $\sigma(x)$ provided by Eq. (5.6) implicitly describes $\sigma(x)$ over an infinite interval of x, whereas we know that it can be valid only over the 9144-m (30,000-ft) interval $(-D/2) < x < (D/2)$, as is indicated in Fig. 16. Thus, we must explicitly include this restriction of the interval size in our series representation.

This can be accomplished by defining

$$\underline{w}(x) \triangleq \begin{cases} w(x) \quad , & (-D/2) < x < (D/2) \\ 0 \quad , & |x| > (D/2) \end{cases} \quad , \tag{5.7}$$

or, equivalently,

$$\underline{w}(x) = \text{rect}\left(\frac{x}{D}\right) w(x) \quad , \tag{5.8}$$

where rect $(\cdot)$ is the rectangular function

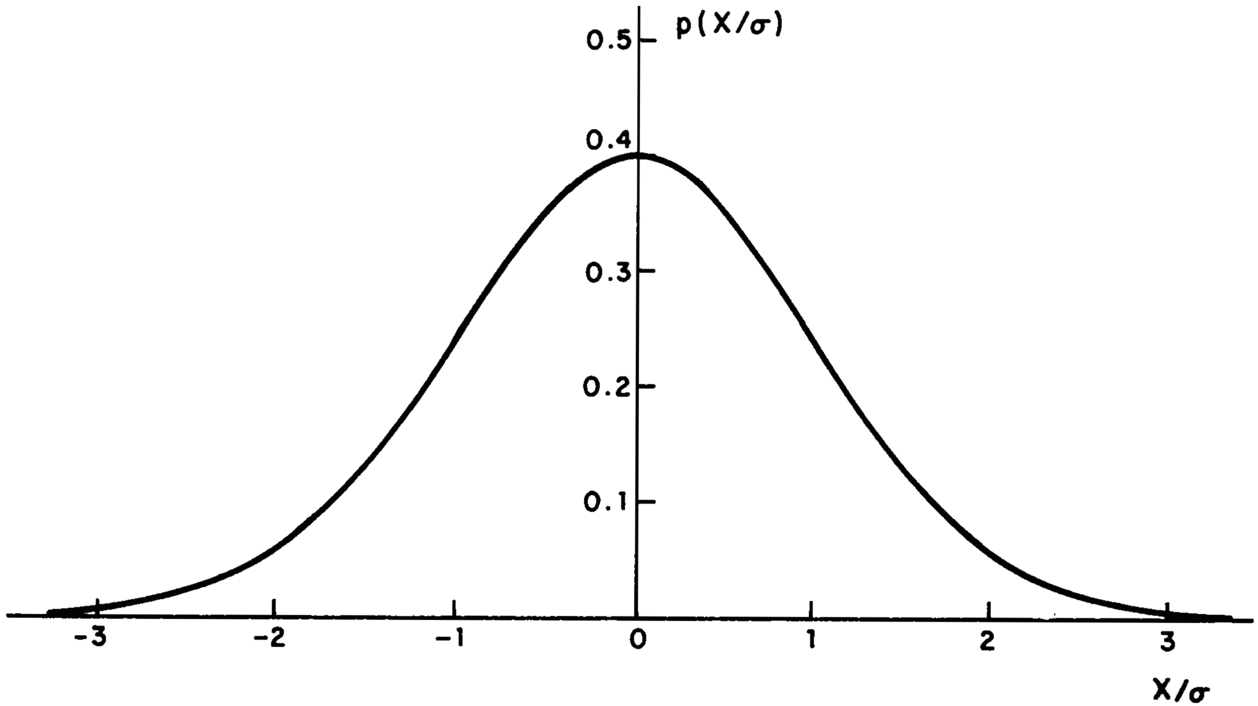

FIG. 15. NORMAL PROBABILITY DENSITY FUNCTION.

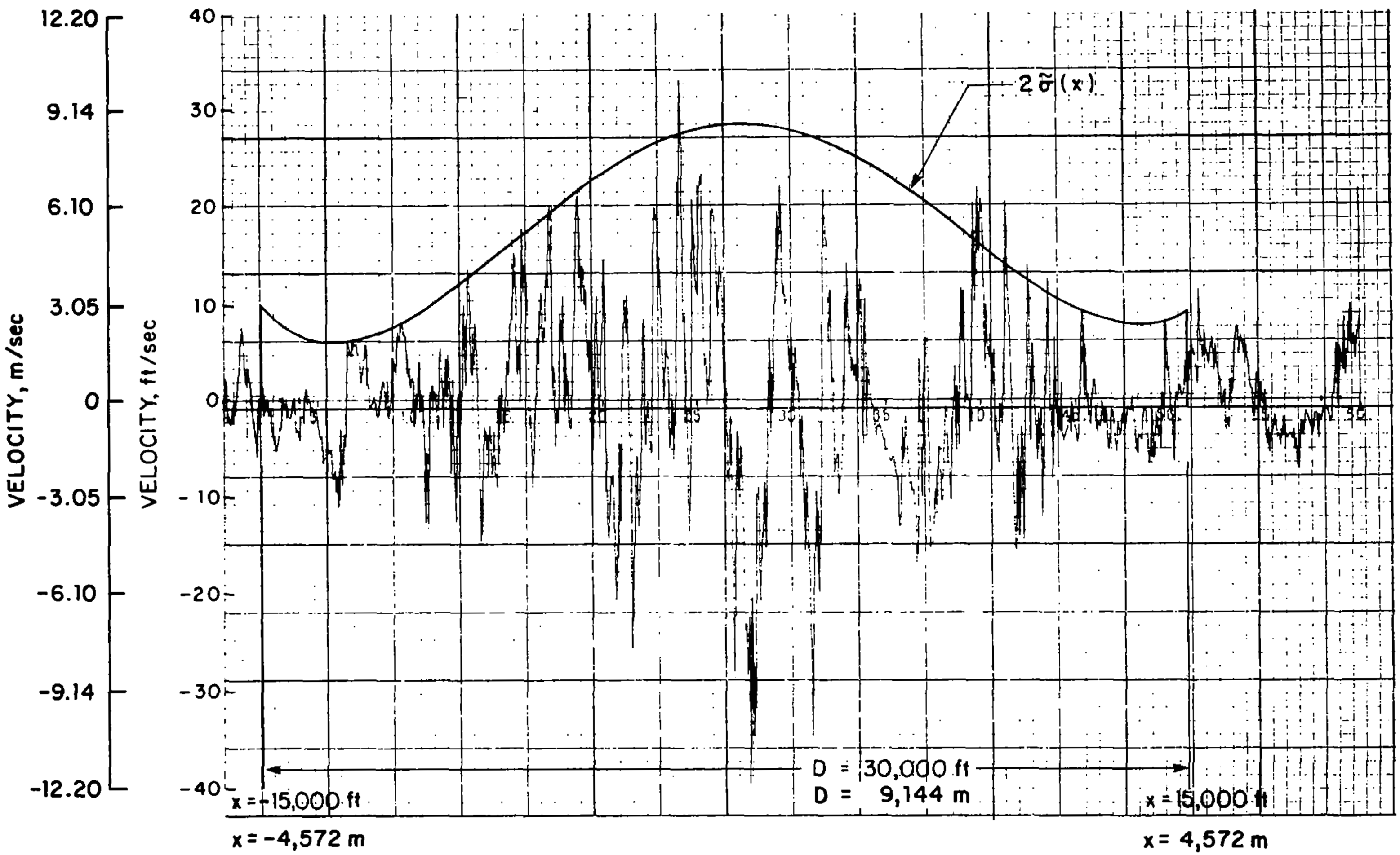

FIG. 16. COMPARISON OF POLYNOMIAL REPRESENTATION $\tilde{\sigma}(x)$ WITH VERTICAL TURBULENCE VELOCITY HISTORY FROM FIG. 12.

$$\mathrm{rect}(\xi) \triangleq \begin{cases} 1 \;, & |\xi| < \frac{1}{2} \\ 0 \;, & |\xi| > \frac{1}{2} \end{cases} \quad . \tag{5.9}$$

The instantaneous autocorrelation function of $\underline{w}(x)$ therefore may be expressed as

$$\begin{aligned} \phi_{\underline{w}}(\xi,x) &= \mathrm{rect}\,(\tfrac{x-\xi/2}{D})\;\mathrm{rect}\,(\tfrac{x+\xi/2}{D})\;\langle w(x-\tfrac{\xi}{2})\;w(x+\tfrac{\xi}{2})\rangle \\ &= \phi_r(\xi,x)\;\phi_w(\xi,x) \quad , \end{aligned} \tag{5.10}$$

where $\phi_w(\xi,x)$ is the instantaneous autocorrelation function of $w(x)$, and where

$$\begin{aligned} \phi_r(\xi,x) &\triangleq \mathrm{rect}\,(\tfrac{x-\xi/2}{D})\;\mathrm{rect}\,(\tfrac{x+\xi/2}{D}) \\ &= \begin{cases} \mathrm{rect}\,[\frac{\xi}{2(D-2x)}] \;, & |x| < \frac{D}{2} \\ 0 \;, & |x| > \frac{D}{2} \end{cases} \quad . \end{aligned} \tag{5.11}$$

Consequently, applying the convolution theorem to Eq. (5.10), we may express the instantaneous spectrum of $\underline{w}(x)$ in terms of the instantaneous spectrum of $w(x)$; i.e.,

$$\Phi_{\underline{w}}(k,x) = \int_{-\infty}^{\infty} \Phi_r(k-\nu,x)\;\Phi_w(\nu,x)d\nu \tag{5.12a}$$

$$= \int_{-\infty}^{\infty} \Phi_r(\nu,x)\;\Phi_w(k-\nu,x)d\nu \quad , \tag{5.12b}$$

where $\Phi_r(\nu,x)$ is the instantaneous spectrum of the rectangular function of width D; i.e.,

$$\Phi_r(k,x) \triangleq \int_{-\infty}^{\infty} \phi_r(\xi,x)e^{-i2\pi k\xi}d\xi \quad . \tag{5.13}$$

Combining Eq. (5.12a) with the series expansion, Eq. (5.1), gives

$$\Phi_{\underline{w}}(k,x) = \sum_{\substack{n=0 \\ n=\text{even}}}^{8} \frac{a_n(x)}{n!} \Phi_{\underline{z}}^{(n)}(k,x) \quad , \tag{5.14}$$

where we have defined

$$\Phi_{\underline{z}}^{(n)}(k,x) \triangleq \int_{-\infty}^{\infty} \Phi_r(k-\nu,x)\ \Phi_z^{(n)}(\nu)d\nu \tag{5.15a}$$

$$= \int_{-\infty}^{\infty} \Phi_r(\nu,x)\ \Phi_z^{(n)}(k-\nu)d\nu \quad . \tag{5.15b}$$

In summary, the change in the series expansion of Eq. (5.1), brought about by the redefinition, Eq. (5.7), of the turbulence process is to replace the expansion functions $\Phi_z^{(n)}(k)$ of the homogeneous component z(x) by the expansion functions defined by Eq. (5.15b). The new expansion functions $\Phi_{\underline{z}}^{(n)}(k,x)$ are functions of x as well as k, whereas the original expansion functions are independent of x. In fact, the new expansion functions are wavenumber-smoothed versions of the original expansion functions. It may be seen from Eqs. (5.11) and (5.15) that for small values of $|x|$ relatively little smoothing of $\Phi_z^{(n)}(k)$ takes place in the operation of Eq. (5.15), whereas when $|x|$ approaches the value of D/2, i.e., the endpoints of the interval over which the expansion is valid, a considerable amount of smoothing of $\Phi_z^{(n)}(k)$ takes place in the operation of Eq. (5.15). The coefficients $a_n(x)$ in Eq. (5.14) are the same for the expansion of $\underline{w}(x)$ as they are in Eq. (5.1), the expansion of w(x).

Expressions for the original expansion functions $\Phi_z^{(n)}(k)$ are provided by Eqs. (4.48b) and (4.63). The expansion functions $\Phi_{\underline{z}}^{(n)}(k,x)$ may be efficiently computed from the $\Phi_z^{(n)}(k)$ by numerically computing the (inverse) Fourier transforms of the original expansion functions, i.e., by first computing

$$\phi_z^{(n)}(\xi) \triangleq \int_{-\infty}^{\infty} \Phi_z^{(n)}(k)e^{i2\pi k\xi}dk \quad . \tag{5.16}$$

According to Eqs. (5.16), (5.15), (5.13) and (5.11), and the convolution theorem, the new expansion functions $\Phi_{\underline{z}}^{(n)}(k,x)$ may then be computed by

$$\Phi_{\underline{z}}^{(n)}(k,x) = \int_{-(D-2x)}^{D-2x} \phi_{z}^{(n)}(\xi)e^{-i2\pi k\xi}d\xi \quad . \tag{5.17}$$

The first term in Eq. (5.14) is no longer a quasi-homogeneous approximation to the instantaneous spectrum because of the smoothing of the homogeneous spectrum described by Eqs. (5.15) and (5.17).

In order to compute the coefficients $a_n(x)$, n = 2,4,6,8 in Eq. (5.14) from the representation for $\sigma(x)$ given by Eq. (5.6) using the formulas for the $a_n(x)$ given in Appendix E, we need expressions for the derivatives of $\tilde{\sigma}(x)$. These derivatives may be obtained by differentiation of Eq. (5.6):

$$\tilde{\sigma}^{(j)}(x) = (\tfrac{2}{D})^{j} \sum_{m=0}^{4} b_m P_m^{(j)}(2x/D) \quad , \tag{5.18}$$

where we have used the "chain rule" and the definition

$$P_m^{(j)}(2x/D) \triangleq \left. \frac{d^j P_m(\xi)}{d\xi^j} \right|_{\xi = 2x/D} \quad . \tag{5.19}$$

Using the notation α = 2/D, we give in Table 1 the expressions for $P_m^{(j)}(\alpha x)$, which were obtained by differentiating the Legendre polynomials of Eq. (5.3).

TABLE 1. TABLE OF EXPRESSIONS FOR $P_m^{(j)}(\alpha x)$

	m=0	m=1	m=2	m=3	m=4
j=0	1	αx	$\frac{1}{2}[3(\alpha x)^2-1]$	$\frac{1}{2}[5(\alpha x)^3-3\alpha x]$	$\frac{1}{8}[35(\alpha x)^4-30(\alpha x)^2+3]$
j=1	0	1	$3\alpha x$	$\frac{1}{2}[15(\alpha x)^2-3]$	$\frac{1}{2}[35(\alpha x)^3-15(\alpha x)]$
j=2	0	0	3	$15\alpha x$	$\frac{1}{2}[105(\alpha x)^2-15]$
j=3	0	0	0	15	$105\alpha x$
j=4	0	0	0	0	105

Evaluation of Instantaneous Spectra of "Slow" Burst of Turbulence

It is evident from Fig. 7 that the maximum values of the magnitudes of the individual terms in Eq. (5.1) occur at wavenumber $k = 0$. According to Eq. (5.1), the value of $\Phi_w(k,x)$, evaluated at $k = 0$, is

$$\Phi_w(0,x) = \sum_{\substack{n=0 \\ n=\text{even}}}^{8} \frac{a_n(x)}{n!} \Phi_z^{(n)}(0)$$

$$= \sum_{\substack{n=0 \\ n=\text{even}}}^{8} \frac{a_n(x)}{n!} L_z^{n+1} \overline{\Phi}_z^{(n)}(0) \quad , \tag{5.20}$$

where we have used Eq. (4.48) in going to the second line. The terms in the right-hand side of Eq. (5.20) have been plotted in Fig. 17 for the case where $\Phi_z(k)$ is the von Karman transverse spectrum with integral scale of L_z = 137.16 m (450 ft) (as was illustrated in Fig. 14). In computing the curves shown in Fig. 17, we used the values of $\overline{\Phi}_z^{(n)}(0)$ given by Eq. (4.65); the values of $a_n(x)$ were computed using the representation of $\sigma(x)$ given by Eq. (5.6) and shown in Fig. 16. The derivatives of $\sigma(x)$ were computed using Eq. (5.18) and Table 1, and the actual evaluation of the $a_n(x)$, $n = 0,2,4,6,8$ from the derivatives was carried out using the expressions for the a's given in Appendix E.

The actual curves plotted in Fig. 17 are 10^n times the individual terms in Eq. (5.20). Consequently, to a first approximation, each term contributes approximately one percent of the preceding term in the neighborhood of $x = 0$, where the maximum value of $\tilde{\sigma}(x)$ occurs, as is shown in Fig. 16. We therefore conclude that, for practical purposes, all correction terms in Eqs. (5.20) and (5.1) are negligible in the present application; thus, we may approximate the instantaneous spectrum of the record shown in Fig. 16 by the quasi-homogeneous approximation

$$\Phi_w(k,x) \approx \sigma^2(x)\, \Phi_z(k) \quad , \tag{5.21}$$

which is the first term of the expansion, Eq. (5.1), since $a_0(x) = \sigma^2(x)$.

This result is completely consistent with the conclusions drawn from our burst of turbulence example discussed in Sec. 4.5

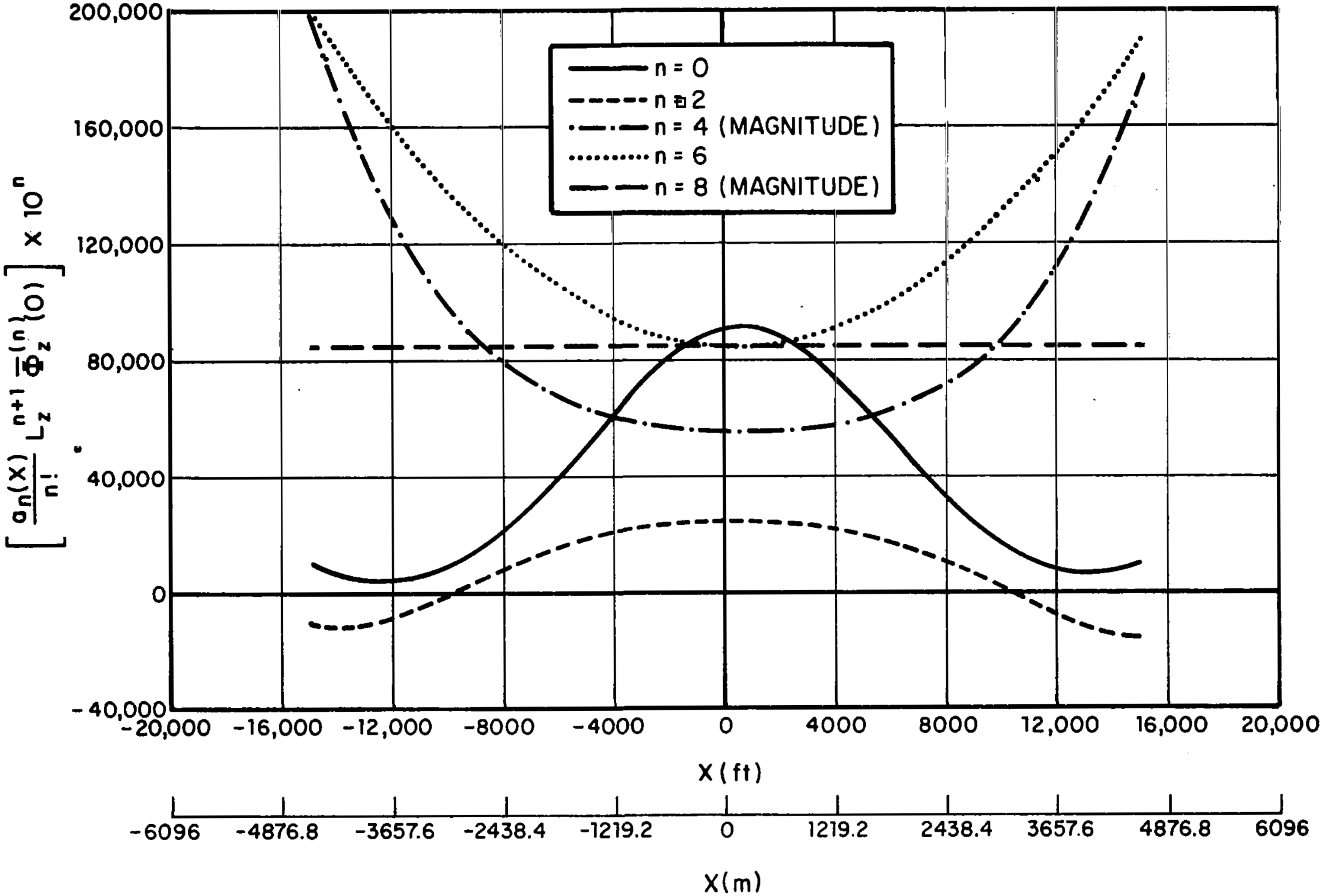

FIG. 17. MAXIMUM VALUES OF TERMS IN EXPANSION OF INSTANTANEOUS SPECTRUM OF NONHOMOGENEOUS VERTICAL RECORD SHOWN IN FIG. 12. (See Eq. 5.20.)

and illustrated in Fig. 10. It was concluded in Sec. 4.5 that when $L_\sigma \geq 13\ L_z$, the nonhomogeneous effects will not be easily detectable in the spectrum. It is evident from Figs. 10 and 16 that a definition of L_σ for the record shown in Fig. 16, equivalent to the definition implicit in Fig. 10, yields for the case shown in Fig. 16, $L_\sigma \approx 4572$ m (15,000 ft) – i.e., one-half of the record length. Consequently, since we determined that $L_z \approx 137.16$ m (450 ft), we have for the record shown in Fig. 16 $(L_\sigma/L_z) \approx (15{,}000/450) = 33.3$, which is much larger than the "threshold of detection" $(L_\sigma/L_z) = 13$. Thus, we could not expect the correction terms in Eq. (5.1) to contribute significantly in the case of the record shown in Fig. 16.

As a consequence of Eq. (5.21), we should expect the usual (energy) spectrum of the nonhomogeneous record shown in Fig. 16 to have the same form as the spectrum obtained by infinite clipping and use of the arcsin law. This (energy) spectrum was computed in the usual way from the 9144-m (30,000-ft) segment shown in Fig. 16. In computing this spectrum, we used the same Papoulis window function as was used in computation of the spectrum shown in Fig. 13. A comparison of the two spectra is shown in Fig. 18, where in both cases, the spectra are normalized to a value of $\sigma^2 = 1$ (since the original record is nonhomogeneous). It is evident from Fig. 18 that no systematic deviation of any consequence occurs between the two spectra. This result is consistent with Eq. (5.21).

Computation of instantaneous spectra of records with length-scaled modulating functions of the shape shown in Fig. 16. By keeping the *shape* of the modulating function $\sigma(x)$ the same, but scaling its length D = 9144 m (30,000 ft) to shorter values, we can determine the various degrees of effect such length-scaled versions of the modulating function shown in Fig. 16 will have on the instantaneous spectra of turbulence records. This investigation has been carried out by using Eq. (5.14) to compute the instantaneous spectra of turbulence records for values of D of 9144, 3048, 1524, and 762 m (30,000, 10,000, 5000, and 2500 ft), for cases where the homogeneous component $z(x)$ of the model of Eq. (3.1) has a von Karman transverse spectrum with an integral scale of L = 137.16 m (450 ft). These calculations used length-scaled versions of the modulating function $\sigma(x)$ illustrated in Fig. 16 and described mathematically by Eq. (5.6).

Comparisons of the instantaneous spectra evaluated by Eq. (5.14) at the value x = 0 (the midpoint of the intervals of length D) are shown in Fig. 19 for the four values of D mentioned above. The spectrum for D = 9144 m (30,000 ft) is indistinguishable from the von Karman transverse spectrum. The spectrum for D = 3048 m (10,000 ft) shows a very slight smoothing of the knee; the spectrum for D = 1524 m (5000 ft) shows a strong smoothing of the knee, and the spectrum for D = 762 m (2500 ft) shows an even stronger smoothing.

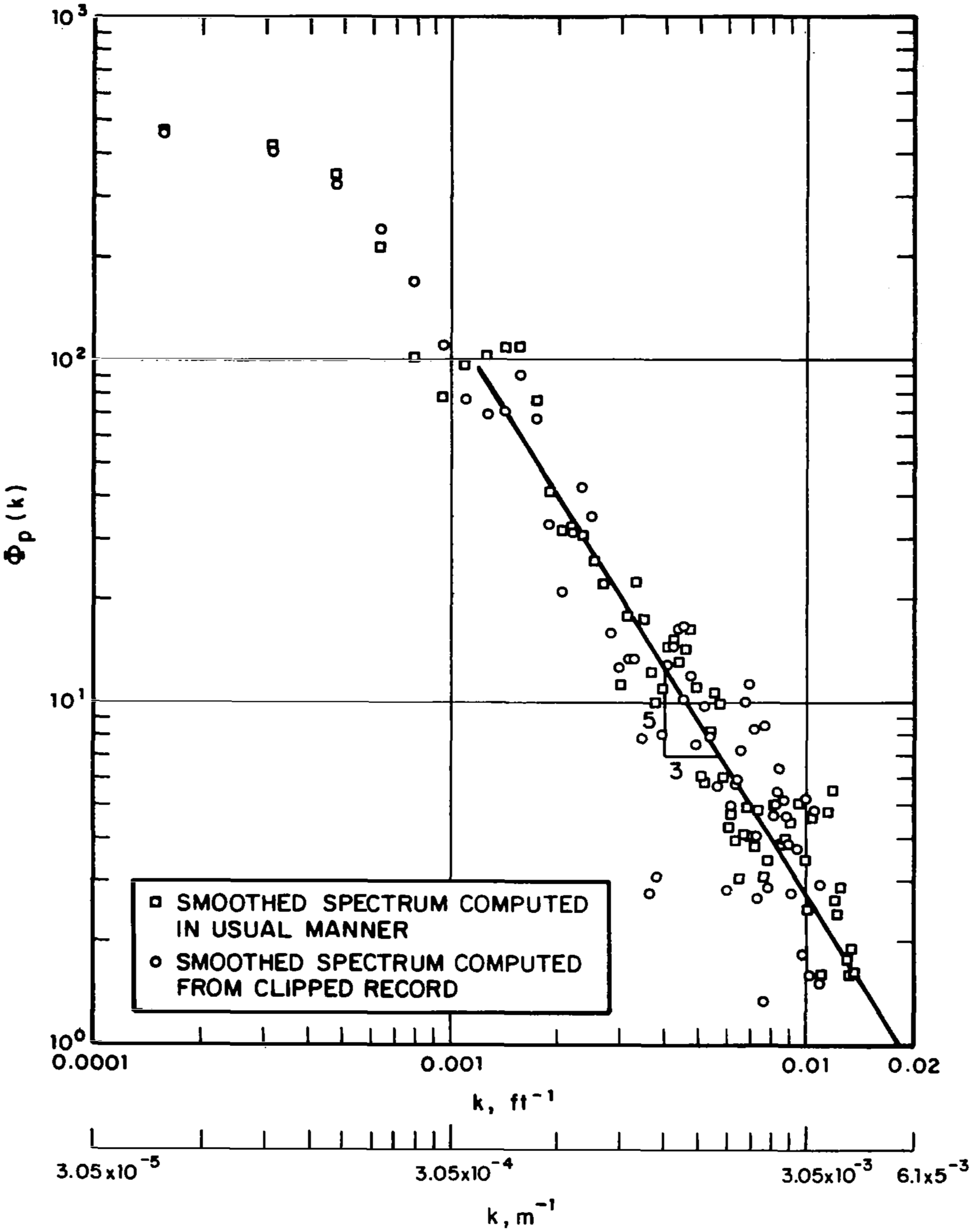

FIG. 18. COMPARISON OF NORMALIZED SPECTRA OF RECORD SHOWN IN FIGS. 12 AND 16. ONE SPECTRUM WAS COMPUTED BY INFINITE CLIPPING AND USE OF THE "ARCSIN LAW." THE OTHER SPECTRUM WAS COMPUTED IN THE USUAL MANNER.

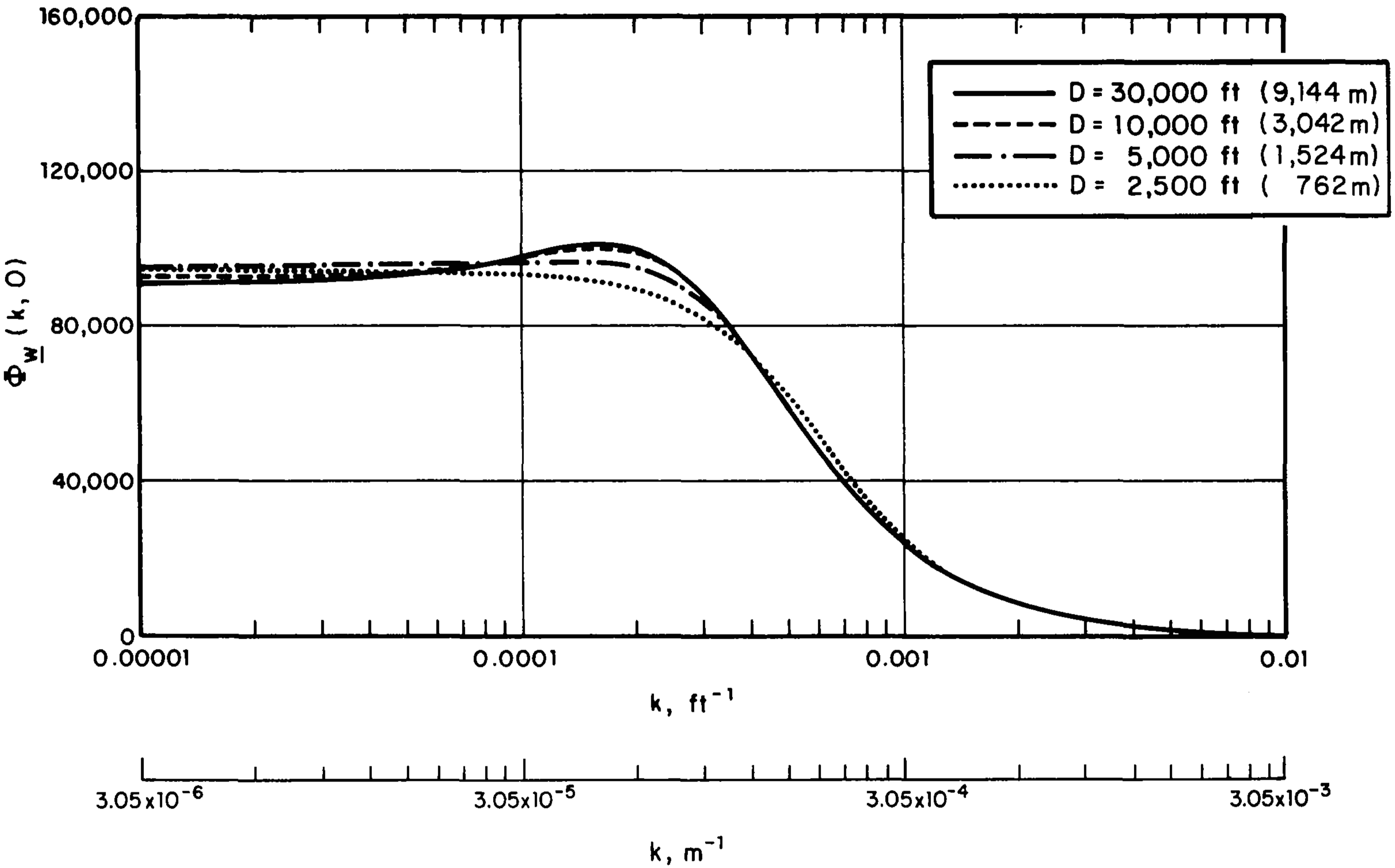

FIG. 19. COMPARISON OF INSTANTANEOUS SPECTRA EVALUATED AT x = 0 FOR VARIOUS LENGTH-SCALED MODULATING FUNCTIONS WITH SAME SHAPE AS THAT SHOWN IN FIG. 16.

The above results are consistent with our conclusions drawn from the burst of turbulence example discussed in Sec. 4.5 and illustrated in Fig. 10. As we mentioned above, a definition of L_σ for the modulating function shown in Fig. 16, equivalent to the definition implicit in Fig. 10, is $L_\sigma \approx D/2$. Thus, using $L_z = 137.16$ m (450 ft), we have for D = 3048 m (10,000 ft) $(L_\sigma/L_z) \approx (5000/450) = 11.1$; for D = 1524 m (5000 ft) $(L_\sigma/L_z) \approx 5.6$; and for D = 762 m (2500 ft) $(L_\sigma/L_z) \approx 2.8$. It was determined in the discussion of Sec. 4.5 that smoothing of the knee should become just detectable for a value of $(L_\sigma/L_z) \approx 13$, whereas strong smoothing of the knee should take place for values of $(L_\sigma/L_z) \leq 7$. The behavior of the spectra shown in Fig. 19 are completely consistent with these results.

The above results further reinforce the conclusions drawn in Sec. 4.6 to the effect that the first and last two bursts of turbulence marked on the vertical record in Fig. 11 would cause strong smoothing of the knee of the von Karman spectrum, whereas, the middle two bursts would cause weak, but detectable, smoothing.

In Fig. 20, the instantaneous spectra computed by Eq. (5.14) for a value of $D = 762$ m (2500 ft) are compared with transverse von Karman spectra for values of x of -190.5, 0, and 190.5 m (-625, 0, and 625 ft). The von Karman spectrum that each of the computed spectra is compared with is the spectrum resulting from the first term in Eq. (5.1), which is the quasi-homogeneous approximation. Each of the spectra shown in Fig. 20 shows appreciable deviation from the von Karman spectrum.

Comparison of Spectra of Flight 32, Run 4 Computed in the Conventional Way and Computed Using Infinite Clipping and the Arcsin Law

The vertical velocity record of NASA Langley Flight 32, Run 4, has a relatively slowly (more or less) monotonically increasing variance. Because of the gradual rate of change of the variance, we did not expect this nonhomogeneous behavior to have any appreciable effect on the shape of the spectrum of the record. This conjecture was verified by computing the spectra of the last 400 sec of the record (where the strongest nonhomogeneous behavior occurred) in two ways. The spectrum was computed in the conventional way, using a Papoulis window function with a value of M = 4807 m (15,770 ft)* (see Appendix B). This conventional spectrum is displayed in Fig. 21. The spectrum was also computed by infinite clipping (after subtraction of the mean value of -0.1988 m/sec (-0.6521 ft/sec) and after correction for the infinite clipping

*Speed of measurement aircraft was 188 m/sec (616 ft/sec).

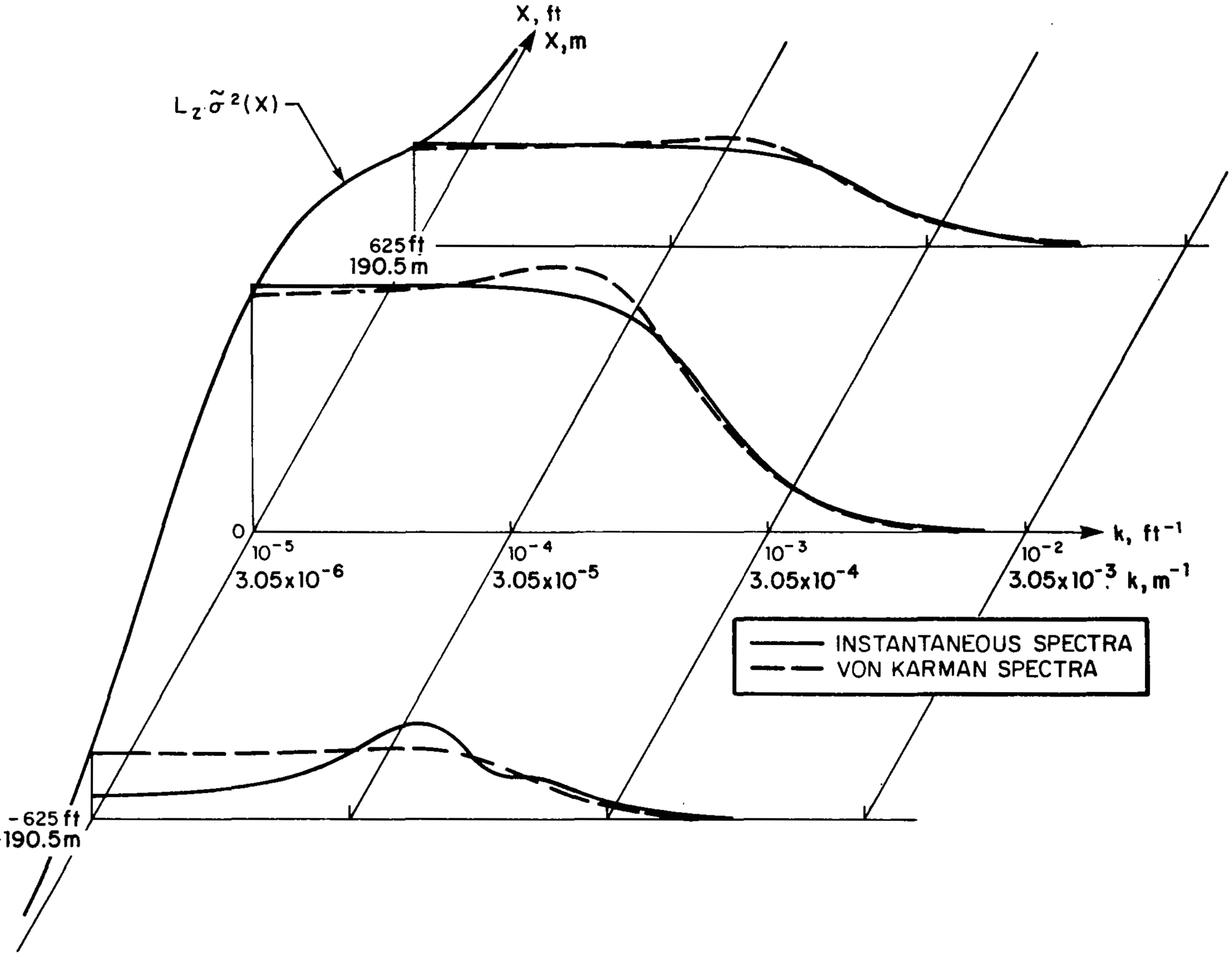

FIG. 20. COMPARISON OF INSTANTANEOUS SPECTRA WITH TRANSVERSE von KARMAN SPECTRA FOR LENGTH-SCALED MODULATING FUNCTION OF SAME SHAPE AS THAT IN FIG. 16 BUT WITH TOTAL LENGTH OF 762 m (2500 ft).

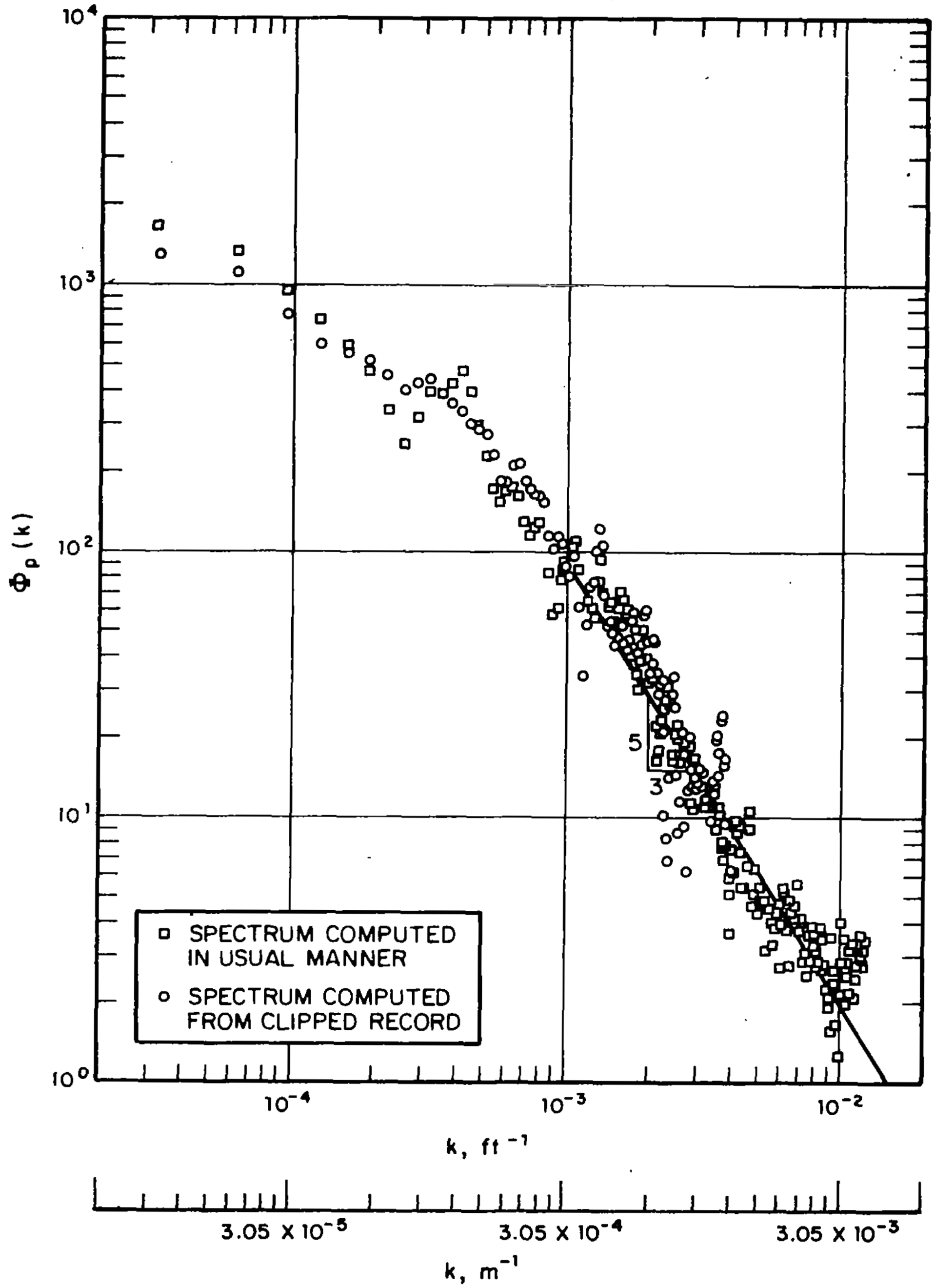

FIG. 21. COMPARISON OF SMOOTHED SPECTRA COMPUTED FROM LAST 400 SECONDS OF VERTICAL VELOCITY COMPONENT OF FLIGHT 32, RUN 4 USING CONVENTIONAL METHOD AND INFINITE CLIPPING CORRECTED BY THE "ARCSIN LAW." SMOOTHING PRODUCES NEGLIGIBLE EFFECT ON SHARPNESS OF THE KNEE. BOTH SPECTRA ARE NORMALIZED TO UNIT VARIANCE.

using the sine transformation of Eq. (3.20). The same Papoulis window function was used in both spectrum calculations.

Since infinite clipping destroys all amplitude information and, therefore, any effects due to the nonhomogeneous variance, it follows that good agreement between the two computed spectra would verify the conjecture that the nonhomogeneous variance has negligible effect on the shape of the spectrum, since the conventional spectrum calculation would retain any effects caused by the nonhomogeneous variance. It may be seen from Fig. 21 that little systematic difference occurs between the two spectra. Consequently, we must conclude that the nonhomogeneous variance of the record has little effect on the shape of the spectrum, as expected. Because the record has a nonhomogeneous variance, both spectra plotted in Fig. 21 have been normalized to unit variance.

In Fig. 22, the two spectra are plotted together with a family of von Karman transverse spectra with integral scales of 121.92, 152.4, 182.88, 243.84, and 304.8 m (400, 500, 600, 800, and 1000 ft). All von Karman spectra shown have unit variance in agreement with the measured spectra. It is evident that the measured spectra, computed in both ways, contain more low-frequency power than any of the von Karman spectra shown and that this deviation from von Karman behavior cannot be attributed to nonhomogeneous behavior of the type described by the uniformly modulated model of Eq. (3.1).

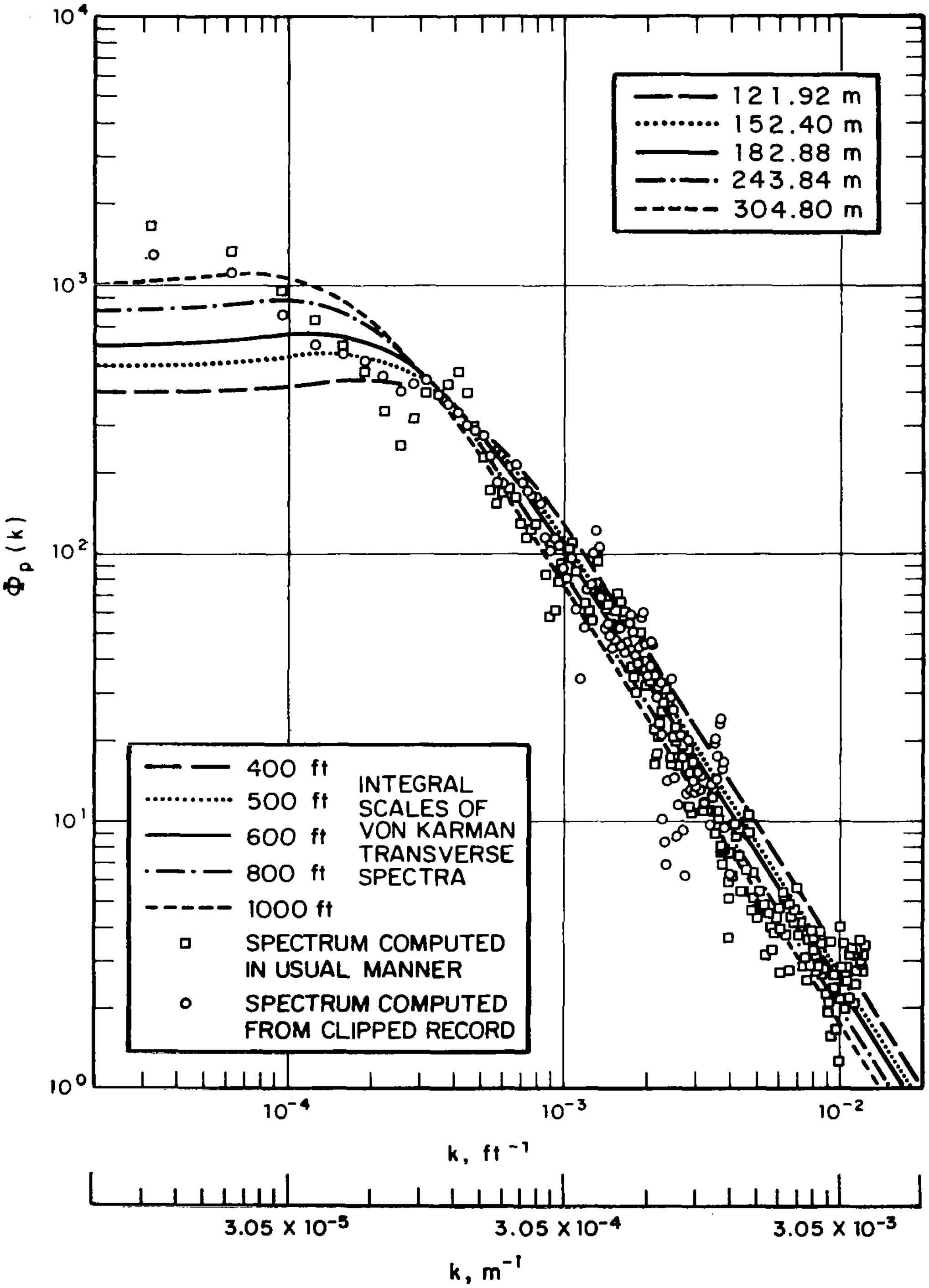

FIG. 22. COMPARISON OF SPECTRA SHOWN IN FIG. 21 WITH von KARMAN TRANSVERSE SPECTRA.

B L A N K P A G E

APPENDIX A

CUMULATIVE PROBABILITY DISTRIBUTION FUNCTION OF VERTICAL VELOCITY COMPONENT OF TEST NO. 190, LEG NO. 5 OF LO-LOCAT PROGRAM

The cumulative probability distribution function P(W) is defined as

$$P(W) \triangleq \int_{-\infty}^{W} p(w)dw \quad , \tag{A.1}$$

where p(w) is the (empirically determined) probability density function. P(W) is tabulated below as a function of velocity W measured in ft/sec. The measured mean value and standard deviation are also listed. These data were computed from the vertical velocity component of Test No. 190, Leg No. 5 of the LO-LOCAT program.

Mean Value = -.039 m/sec (-0.128 ft/sec)

Standard Deviation = 2.94 m/sec (9.63 ft/sec)

W	P(W)
-49,9	0,0000
-49,8	0,0000
-49,7	0,0000
-49,6	0,0000
-49,5	0,0000
-49,4	0,0000
-49,3	0,0000
-49,2	0,0000
-49,1	0,0000
-49,0	0,0000
-48,9	0,0000
-48,8	0,0000
-48,7	0,0000
-48,6	0,0000
-48,5	0,0000
-48,4	0,0000
-48,3	0,0000
-48,2	0,0000
-48,1	0,0000
-48,0	0,0000
-47,9	0,0000
-47,8	0,0000
-47,7	0,0000
-47,6	0,0000
-47,5	0,0000
-47,4	0,0000
-47,3	0,0000
-47,2	0,0000
-47,1	0,0000
-47,0	0,0000
-46,9	0,0000
-46,8	0,0000
-46,7	0,0000
-46,6	0,0000
-46,5	0,0000
-46,4	0,0000
-46,3	0,0000
-46,2	0,0000
-46,1	0,0000
-46,0	0,0000
-45,9	0,0000
-45,8	0,0000
-45,7	0,0000
-45,6	0,0000
-45,5	0,0000
-45,4	0,0000
-45,3	0,0000
-45,2	0,0000
-45,1	0,0000
-45,0	0,0000
-44,9	0,0000
-44,8	0,0000
-44,7	0,0000
-44,6	0,0000
-44,5	0,0000
-44,4	0,0000
-44,3	0,0000
-44,2	0,0000
-44,1	0,0000
-44,0	0,0000
-43,9	0,0000
-43,8	0,0000
-43,7	0,0000
-43,6	0,0000
-43,5	0,0000
-43,4	0,0000
-43,3	0,0000
-43,2	0,0000
-43,1	0,0000
-43,0	0,0000
-42,9	0,0000
-42,8	0,0000
-42,7	0,0000
-42,6	0,0000
-42,5	0,0000
-42,4	0,0000
-42,3	0,0000
-42,2	0,0000
-42,1	0,0000
-42,0	0,0000
-41,9	0,0000
-41,8	0,0000
-41,7	0,0001
-41,6	0,0001
-41,5	0,0001
-41,4	0,0001
-41,3	0,0001
-41,2	0,0001

W	P(W)
-41,1	0,0001
-41,0	0,0001
-40,9	0,0001
-40,8	0,0001
-40,7	0,0001
-40,6	0,0001
-40,5	0,0001
-40,4	0,0001
-40,3	0,0001
-40,2	0,0001
-40,1	0,0001
-40,0	0,0002
-39,9	0,0002
-39,8	0,0002
-39,7	0,0002
-39,6	0,0002
-39,5	0,0003
-39,4	0,0003
-39,3	0,0003
-39,2	0,0003
-39,1	0,0003
-39,0	0,0003
-38,9	0,0003
-38,8	0,0003
-38,7	0,0004
-38,6	0,0004
-38,5	0,0004
-38,4	0,0004
-38,3	0,0004
-38,2	0,0004
-38,1	0,0004
-38,0	0,0004
-37,9	0,0004
-37,8	0,0005
-37,7	0,0005
-37,6	0,0005
-37,5	0,0005
-37,4	0,0005
-37,3	0,0005
-37,2	0,0005
-37,1	0,0005
-37,0	0,0006
-36,9	0,0006
-36,8	0,0007
-36,7	0,0007
-36,6	0,0007
-36,5	0,0008
-36,4	0,0008
-36,3	0,0008
-36,2	0,0008
-36,1	0,0009
-36,0	0,0009
-35,9	0,0010
-35,8	0,0010
-35,7	0,0010
-35,6	0,0010
-35,5	0,0011
-35,4	0,0011
-35,3	0,0011
-35,2	0,0011
-35,1	0,0012
-35,0	0,0012
-34,9	0,0012
-34,8	0,0012
-34,7	0,0012
-34,6	0,0012
-34,5	0,0012
-34,4	0,0013
-34,3	0,0014
-34,2	0,0014
-34,1	0,0014
-34,0	0,0014
-33,9	0,0014
-33,8	0,0014
-33,7	0,0014
-33,6	0,0015
-33,5	0,0015
-33,4	0,0016
-33,3	0,0016
-33,2	0,0016
-33,1	0,0016
-33,0	0,0017
-32,9	0,0018
-32,8	0,0019
-32,7	0,0019
-32,6	0,0020
-32,5	0,0021
-32,4	0,0022
-32,3	0,0022
-32,2	0,0023
-32,1	0,0023
-32,0	0,0023
-31,9	0,0024
-31,8	0,0024
-31,7	0,0024
-31,6	0,0024
-31,5	0,0025
-31,4	0,0026
-31,3	0,0026
-31,2	0,0026
-31,1	0,0027
-31,0	0,0027

W	P(W)
-30,9	0,0028
-30,8	0,0029
-30,7	0,0030
-30,6	0,0030
-30,5	0,0031
-30,4	0,0031
-30,3	0,0031
-30,2	0,0031
-30,1	0,0032
-30,0	0,0032
-29,9	0,0032
-29,8	0,0033
-29,7	0,0033
-29,6	0,0034
-29,5	0,0035
-29,4	0,0036
-29,3	0,0036
-29,2	0,0036
-29,1	0,0037
-29,0	0,0039
-28,9	0,0040
-28,8	0,0042
-28,7	0,0042
-28,6	0,0043
-28,5	0,0044
-28,4	0,0044
-28,3	0,0045
-28,2	0,0045
-28,1	0,0046
-28,0	0,0046
-27,9	0,0046
-27,8	0,0047
-27,7	0,0048
-27,6	0,0048
-27,5	0,0049
-27,4	0,0049
-27,3	0,0050
-27,2	0,0050
-27,1	0,0051
-27,0	0,0051
-26,9	0,0052
-26,8	0,0052
-26,7	0,0053
-26,6	0,0053
-26,5	0,0054
-26,4	0,0054
-26,3	0,0056
-26,2	0,0056
-26,1	0,0057
-26,0	0,0057
-25,9	0,0058
-25,8	0,0059
-25,7	0,0060
-25,6	0,0061
-25,5	0,0063
-25,4	0,0066
-25,3	0,0068
-25,2	0,0070
-25,1	0,0071
-25,0	0,0072
-24,9	0,0073
-24,8	0,0073
-24,7	0,0076
-24,6	0,0077
-24,5	0,0080
-24,4	0,0081
-24,3	0,0082
-24,2	0,0085
-24,1	0,0086
-24,0	0,0087
-23,9	0,0090
-23,8	0,0091
-23,7	0,0093
-23,6	0,0097
-23,5	0,0098
-23,4	0,0101
-23,3	0,0103
-23,2	0,0104
-23,1	0,0106
-23,0	0,0108
-22,9	0,0109
-22,8	0,0112
-22,7	0,0114
-22,6	0,0117
-22,5	0,0119
-22,4	0,0122
-22,3	0,0124
-22,2	0,0127
-22,1	0,0129
-22,0	0,0132
-21,9	0,0136
-21,8	0,0138
-21,7	0,0141
-21,6	0,0144
-21,5	0,0146
-21,4	0,0147
-21,3	0,0152
-21,2	0,0156
-21,1	0,0159
-21,0	0,0165
-20,9	0,0168
-20,8	0,0172

W	P(W)
-20,7	0,0175
-20,6	0,0177
-20,5	0,0180
-20,4	0,0186
-20,3	0,0189
-20,2	0,0192
-20,1	0,0195
-20,0	0,0200
-19,9	0,0205
-19,8	0,0206
-19,7	0,0213
-19,6	0,0218
-19,5	0,0223
-19,4	0,0232
-19,3	0,0235
-19,2	0,0239
-19,1	0,0244
-19,0	0,0250
-18,9	0,0256
-18,8	0,0262
-18,7	0,0268
-18,6	0,0277
-18,5	0,0282
-18,4	0,0289
-18,3	0,0296
-18,2	0,0304
-18,1	0,0314
-18,0	0,0319
-17,9	0,0327
-17,8	0,0334
-17,7	0,0341
-17,6	0,0348
-17,5	0,0356
-17,4	0,0365
-17,3	0,0373
-17,2	0,0378
-17,1	0,0387
-17,0	0,0394
-16,9	0,0398
-16,8	0,0403
-16,7	0,0416
-16,6	0,0424
-16,5	0,0433
-16,4	0,0441
-16,3	0,0446
-16,2	0,0453
-16,1	0,0461
-16,0	0,0471
-15,9	0,0479
-15,8	0,0488
-15,7	0,0494
-15,6	0,0503
-15,5	0,0518
-15,4	0,0529
-15,3	0,0540
-15,2	0,0550
-15,1	0,0563
-15,0	0,0575
-14,9	0,0587
-14,8	0,0596
-14,7	0,0608
-14,6	0,0618
-14,5	0,0632
-14,4	0,0649
-14,3	0,0663
-14,2	0,0677
-14,1	0,0690
-14,0	0,0705
-13,9	0,0720
-13,8	0,0739
-13,7	0,0759
-13,6	0,0775
-13,5	0,0793
-13,4	0,0805
-13,3	0,0824
-13,2	0,0839
-13,1	0,0856
-13,0	0,0874
-12,9	0,0894
-12,8	0,0909
-12,7	0,0929
-12,6	0,0947
-12,5	0,0962
-12,4	0,0982
-12,3	0,0996
-12,2	0,1014
-12,1	0,1029
-12,0	0,1048
-11,9	0,1065
-11,8	0,1082
-11,7	0,1102
-11,6	0,1119
-11,5	0,1138
-11,4	0,1159
-11,3	0,1177
-11,2	0,1201
-11,1	0,1217
-11,0	0,1237
-10,9	0,1257
-10,8	0,1282
-10,7	0,1300
-10,6	0,1324

W	P(W)
-10,5	0,1354
-10,4	0,1379
-10,3	0,1398
-10,2	0,1424
-10,1	0,1445
-10,0	0,1471
-9,9	0,1498
-9,8	0,1527
-9,7	0,1549
-9,6	0,1573
-9,5	0,1597
-9,4	0,1629
-9,3	0,1651
-9,2	0,1675
-9,1	0,1700
-9,0	0,1727
-8,9	0,1750
-8,8	0,1777
-8,7	0,1803
-8,6	0,1837
-8,5	0,1862
-8,4	0,1889
-8,3	0,1921
-8,2	0,1954
-8,1	0,1985
-8,0	0,2006
-7,9	0,2027
-7,8	0,2050
-7,7	0,2077
-7,6	0,2111
-7,5	0,2137
-7,4	0,2164
-7,3	0,2198
-7,2	0,2239
-7,1	0,2269
-7,0	0,2295
-6,9	0,2323
-6,8	0,2354
-6,7	0,2383
-6,6	0,2415
-6,5	0,2446
-6,4	0,2476
-6,3	0,2520
-6,2	0,2551
-6,1	0,2578
-6,0	0,2613
-5,9	0,2656
-5,8	0,2692
-5,7	0,2726
-5,6	0,2760
-5,5	0,2795
-5,4	0,2828
-5,3	0,2864
-5,2	0,2901
-5,1	0,2939
-5,0	0,2992
-4,9	0,3016
-4,8	0,3052
-4,7	0,3087
-4,6	0,3120
-4,5	0,3161
-4,4	0,3203
-4,3	0,3245
-4,2	0,3284
-4,1	0,3325
-4,0	0,3365
-3,9	0,3416
-3,8	0,3464
-3,7	0,3504
-3,6	0,3545
-3,5	0,3583
-3,4	0,3627
-3,3	0,3665
-3,2	0,3713
-3,1	0,3756
-3,0	0,3789
-2,9	0,3840
-2,8	0,3881
-2,7	0,3920
-2,6	0,3966
-2,5	0,4005
-2,4	0,4051
-2,3	0,4089
-2,2	0,4132
-2,1	0,4173
-2,0	0,4212
-1,9	0,4257
-1,8	0,4309
-1,7	0,4359
-1,6	0,4400
-1,5	0,4439
-1,4	0,4478
-1,3	0,4517
-1,2	0,4553
-1,1	0,4592
-1,0	0,4637
-0,9	0,4681
-0,8	0,4730
-0,7	0,4774
-0,6	0,4814
-0,5	0,4861
-0,4	0,4899
-0,3	0,4932
-0,2	0,4977
-0,1	0,5010
-0,0	0,5065

W	P(W)
0.1	0.5129
0.2	0.5171
0.3	0.5214
0.4	0.5257
0.5	0.5300
0.6	0.5336
0.7	0.5377
0.8	0.5419
0.9	0.5473
1.0	0.5512
1.1	0.5560
1.2	0.5597
1.3	0.5633
1.4	0.5670
1.5	0.5712
1.6	0.5756
1.7	0.5793
1.8	0.5835
1.9	0.5881
2.0	0.5926
2.1	0.5964
2.2	0.6005
2.3	0.6041
2.4	0.6088
2.5	0.6120
2.6	0.6168
2.7	0.6215
2.8	0.6252
2.9	0.6292
3.0	0.6320
3.1	0.6365
3.2	0.6403
3.3	0.6437
3.4	0.6477
3.5	0.6512
3.6	0.6550
3.7	0.6592
3.8	0.6620
3.9	0.6668
4.0	0.6709
4.1	0.6746
4.2	0.6781
4.3	0.6819
4.4	0.6853
4.5	0.6898
4.6	0.6927
4.7	0.6958
4.8	0.6995
4.9	0.7028
5.0	0.7057
5.1	0.7093
5.2	0.7127
5.3	0.7162
5.4	0.7198
5.5	0.7229
5.6	0.7263
5.7	0.7302
5.8	0.7336
5.9	0.7366
6.0	0.7401
6.1	0.7438
6.2	0.7471
6.3	0.7503
6.4	0.7536
6.5	0.7568
6.6	0.7595
6.7	0.7620
6.8	0.7656
6.9	0.7679
7.0	0.7717
7.1	0.7752
7.2	0.7786
7.3	0.7811
7.4	0.7841
7.5	0.7871
7.6	0.7896
7.7	0.7931
7.8	0.7962
7.9	0.7991
8.0	0.8026
8.1	0.8064
8.2	0.8089
8.3	0.8118
8.4	0.8140
8.5	0.8172
8.6	0.8201
8.7	0.8227
8.8	0.8263

W	P(W)
8.9	0.8287
9.0	0.8314
9.1	0.8339
9.2	0.8371
9.3	0.8400
9.4	0.8430
9.5	0.8458
9.6	0.8486
9.7	0.8515
9.8	0.8538
9.9	0.8561
10.0	0.8591
10.1	0.8617
10.2	0.8639
10.3	0.8656
10.4	0.8680
10.5	0.8701
10.6	0.8722
10.7	0.8742
10.8	0.8762
10.9	0.8781
11.0	0.8802
11.1	0.8819
11.2	0.8832
11.3	0.8852
11.4	0.8878
11.5	0.8901
11.6	0.8919
11.7	0.8940
11.8	0.8959
11.9	0.8980
12.0	0.8994
12.1	0.9015
12.2	0.9035
12.3	0.9052
12.4	0.9066
12.5	0.9084
12.6	0.9100
12.7	0.9113
12.8	0.9129
12.9	0.9144
13.0	0.9160
13.1	0.9175
13.2	0.9189
13.3	0.9206
13.4	0.9222
13.5	0.9236
13.6	0.9249
13.7	0.9261
13.8	0.9277
13.9	0.9290
14.0	0.9303
14.1	0.9319
14.2	0.9329
14.3	0.9338
14.4	0.9351
14.5	0.9364
14.6	0.9376
14.7	0.9391
14.8	0.9403
14.9	0.9413
15.0	0.9426
15.1	0.9436
15.2	0.9446
15.3	0.9455
15.4	0.9469
15.5	0.9483
15.6	0.9494
15.7	0.9500
15.8	0.9513
15.9	0.9522
16.0	0.9529
16.1	0.9534
16.2	0.9544
16.3	0.9552
16.4	0.9562
16.5	0.9570
16.6	0.9580
16.7	0.9588
16.8	0.9599
16.9	0.9609
17.0	0.9615
17.1	0.9622
17.2	0.9631
17.3	0.9641
17.4	0.9648
17.5	0.9657
17.6	0.9665
17.7	0.9672
17.8	0.9683
17.9	0.9691
18.0	0.9695
18.1	0.9702
18.2	0.9708
18.3	0.9714
18.4	0.9721
18.5	0.9728
18.6	0.9732
18.7	0.9739
18.8	0.9744
18.9	0.9751
19.0	0.9760

W	P(W)
19.1	0.9765
19.2	0.9772
19.3	0.9778
19.4	0.9784
19.5	0.9790
19.6	0.9796
19.7	0.9798
19.8	0.9804
19.9	0.9810
20.0	0.9815
20.1	0.9819
20.2	0.9823
20.3	0.9828
20.4	0.9831
20.5	0.9834
20.6	0.9838
20.7	0.9842
20.8	0.9844
20.9	0.9847
21.0	0.9851
21.1	0.9853
21.2	0.9858
21.3	0.9861
21.4	0.9865
21.5	0.9869
21.6	0.9873
21.7	0.9875
21.8	0.9878
21.9	0.9880
22.0	0.9883
22.1	0.9887
22.2	0.9890
22.3	0.9893
22.4	0.9897
22.5	0.9900
22.6	0.9904
22.7	0.9905
22.8	0.9907
22.9	0.9910
23.0	0.9911
23.1	0.9913
23.2	0.9915
23.3	0.9917
23.4	0.9919
23.5	0.9923
23.6	0.9926
23.7	0.9928
23.8	0.9930
23.9	0.9932
24.0	0.9935
24.1	0.9937
24.2	0.9938
24.3	0.9940
24.4	0.9943
24.5	0.9944
24.6	0.9946
24.7	0.9948
24.8	0.9950
24.9	0.9953
25.0	0.9955
25.1	0.9956
25.2	0.9958
25.3	0.9961
25.4	0.9961
25.5	0.9963
25.6	0.9964
25.7	0.9966
25.8	0.9966
25.9	0.9967
26.0	0.9968
26.1	0.9969
26.2	0.9970
26.3	0.9972
26.4	0.9973
26.5	0.9973
26.6	0.9974
26.7	0.9974
26.8	0.9975
26.9	0.9975
27.0	0.9975
27.1	0.9976
27.2	0.9977
27.3	0.9977
27.4	0.9979
27.5	0.9979
27.6	0.9980
27.7	0.9981
27.8	0.9982
27.9	0.9983
28.0	0.9984
28.1	0.9984
28.2	0.9984
28.3	0.9983
28.4	0.9985
28.5	0.9985
28.6	0.9986
28.7	0.9987
28.8	0.9988
28.9	0.9989
29.0	0.9989
29.1	0.9989
29.2	0.9989

W	P(W)
29.3	0.9989
29.4	0.9989
29.5	0.9989
29.6	0.9989
29.7	0.9989
29.8	0.9989
29.9	0.9989
30.0	0.9989
30.1	0.9989
30.2	0.9990
30.3	0.9990
30.4	0.9990
30.5	0.9991
30.6	0.9991
30.7	0.9991
30.8	0.9991
30.9	0.9992
31.0	0.9992
31.1	0.9993
31.2	0.9993
31.3	0.9993
31.4	0.9993
31.5	0.9993
31.6	0.9993
31.7	0.9994
31.8	0.9994
31.9	0.9994
32.0	0.9995
32.1	0.9995
32.2	0.9996
32.3	0.9996
32.4	0.9996
32.5	0.9996
32.6	0.9997
32.7	0.9997
32.8	0.9997
32.9	0.9997
33.0	0.9997
33.1	0.9997
33.2	0.9997
33.3	0.9997
33.4	0.9998
33.5	0.9998
33.6	0.9998
33.7	0.9998
33.8	0.9998
33.9	0.9998
34.0	0.9998
34.1	0.9998
34.2	0.9998
34.3	0.9998
34.4	0.9998
34.5	0.9998
34.6	0.9999
34.7	0.9999
34.8	0.9999
34.9	0.9999
35.0	0.9999
35.1	0.9999
35.2	0.9999
35.3	0.9999
35.4	0.9999
35.5	0.9999
35.6	0.9999
35.7	0.9999
35.8	0.9999
35.9	0.9999
36.0	0.9999
36.1	0.9999
36.2	0.9999
36.3	0.9999
36.4	0.9999
36.5	0.9999
36.6	0.9999
36.7	0.9999
36.8	0.9999
36.9	0.9999
37.0	0.9999
37.1	1.0000
37.2	1.0000
37.3	1.0000
37.4	1.0000
37.5	1.0000
37.6	1.0000
37.7	1.0000
37.8	1.0000
37.9	1.0000
38.0	1.0000
38.1	1.0000
38.2	1.0000
38.3	1.0000
38.4	1.0000
38.5	1.0000
38.6	1.0000
38.7	1.0000
38.8	1.0000
38.9	1.0000
39.0	1.0000
39.1	1.0000
39.2	1.0000
39.3	1.0000
39.4	1.0000

W	P(W)
39.5	1.0000
39.6	1.0000
39.7	1.0000
39.8	1.0000
39.9	1.0000
40.0	1.0000
40.1	1.0000
40.2	1.0000
40.3	1.0000
40.4	1.0000
40.5	1.0000
40.6	1.0000
40.7	1.0000
40.8	1.0000
40.9	1.0000
41.0	1.0000
41.1	1.0000
41.2	1.0000
41.3	1.0000
41.4	1.0000
41.5	1.0000
41.6	1.0000
41.7	1.0000
41.8	1.0000
41.9	1.0000
42.0	1.0000
42.1	1.0000
42.2	1.0000
42.3	1.0000
42.4	1.0000
42.5	1.0000
42.6	1.0000
42.7	1.0000
42.8	1.0000
42.9	1.0000
43.0	1.0000
43.1	1.0000
43.2	1.0000
43.3	1.0000
43.4	1.0000
43.5	1.0000
43.6	1.0000
43.7	1.0000
43.8	1.0000
43.9	1.0000
44.0	1.0000
44.1	1.0000
44.2	1.0000
44.3	1.0000
44.4	1.0000
44.5	1.0000
44.6	1.0000
44.7	1.0000
44.8	1.0000
44.9	1.0000
45.0	1.0000
45.1	1.0000
45.2	1.0000
45.3	1.0000
45.4	1.0000
45.5	1.0000
45.6	1.0000
45.7	1.0000
45.8	1.0000
45.9	1.0000
46.0	1.0000
46.1	1.0000
46.2	1.0000
46.3	1.0000
46.4	1.0000
46.5	1.0000
46.6	1.0000
46.7	1.0000
46.8	1.0000
46.9	1.0000
47.0	1.0000
47.1	1.0000
47.2	1.0000
47.3	1.0000
47.4	1.0000
47.5	1.0000
47.6	1.0000
47.7	1.0000
47.8	1.0000
47.9	1.0000
48.0	1.0000
48.1	1.0000
48.2	1.0000
48.3	1.0000
48.4	1.0000
48.5	1.0000
48.6	1.0000
48.7	1.0000
48.8	1.0000
48.9	1.0000
49.0	1.0000
49.1	1.0000
49.2	1.0000
49.3	1.0000
49.4	1.0000
49.5	1.0000
49.6	1.0000
49.7	1.0000
49.8	1.0000
49.9	1.0000
50.0	1.0000

APPENDIX B

COMPUTATIONAL PROCEDURE FOR DETERMINATION OF SPECTRA OF HOMOGENEOUS SAMPLE

Conventional Power Spectrum Calculation

The turbulence velocity records were recorded as functions of time t. We wished to compute wavenumber spectra; hence, time coordinate t was converted to spatial coordinate x by invoking Taylor's hypothesis, i.e., $x = Vt$, where V is the speed of the aircraft carrying out the measurements.

The Fourier transform of the frequency smoothing function used was [*15*]

$$p_0(\xi) = \begin{cases} \frac{1}{\pi}\left|\sin\frac{\pi\xi}{M}\right| + \left(1 - \frac{|\xi|}{M}\right)\cos\frac{\pi\xi}{M} , & |\xi| \leq M \\ 0 , & |\xi| > M . \end{cases} \tag{B.1}$$

Notice that $p_0(\xi)$ has a total length of 2M meters. Consequently, to prevent aliasing errors, it was necessary to add M "meters of zeros" to our original record.

Denote the actual length of the record by $2\ell-M$ meters. Then, after the addition of the M meters of zeros, our total record length was 2ℓ meters. The first step in the computation was to form the (unsmoothed) wavenumber spectrum of the record w(x):

$$\Phi_\ell(k) = \frac{1}{2\ell-M}\left|\int_0^{2\ell} w(x)e^{-i2\pi kx}dx\right|^2 , \tag{B.2}$$

where, as noted above, w(x), as used in Eq. (B.2), is identically zero in the interval $(2\ell-M) < x < 2\ell$. Once $\Phi_\ell(k)$ was computed, its (fast) inverse Fourier transform was computed, which yielded the sample autocorrelation function

$$R_\ell(\xi) = \int \Phi_\ell(k)e^{i2\pi kx}dk . \tag{B.3}$$

Smoothing (i.e., convolution) in the wavenumber domain is equivalent to multiplication in its transform domain. Consequently, the next step was to multiply $R_\ell(\xi)$ by the transform, Eq. (B.1), of the frequency smoothing function and then to (fast) Fourier transform the resulting product, after correction of $R_\ell(\xi)$, for the finite length (2ℓ-M) of the original sample. The resulting transform is

$$\Phi_p(k) = \int_{-M}^{M} p_0(\xi) \left[\frac{R_\ell(\xi)}{1 - \frac{|\xi|}{2\ell - M}} \right] e^{-i2\pi k\xi} d\xi \quad . \tag{B.4}$$

$\Phi_p(k)$ is our frequency-smoothed spectrum computed in the "conventional way", and corrected for finite sample length.

In computing the conventional spectrum displayed in Fig. 4, a value of M = 932.1 m (3058 ft) was used in the window function of Eq. (B.1). This value corresponded to 512 samples of the record and to 5.12 sec of the original record, since the sampling rate was 100 samples/sec. The measurement aircraft speed was 182 m/sec (597.3 ft/sec), and the total duration of the record was 270 sec, which yielded a total of 27,000 samples used in the transform of Eq. (B.2).

Power Spectrum Calculation From Infinitely Clipped Record Using "Arcsin Law" Correction

The same record of length 2ℓ-M meters was used to compute the power spectrum using "infinite clipping" and the "arcsin law". The first step was to subtract the mean value of m = -0.039 m/sec (-0.128 ft/sec) from the record. The sign of each of the corrected samples was then examined; positive samples were assigned a value of +1 and negative samples were assigned a value of -1. Any sample identically equal to zero was given the value of the preceeding sample. Let us call the resulting waveform z(x). M "meters of zeros" was then added to z(x) and resulted in a record of total length equal to 2ℓ meters. The sample spectrum $\Phi_{\ell z}(k)$ of the resulting record was then computed:

$$\Phi_{\ell z}(k) = \frac{1}{2\ell - M} \left| \int_0^{2\ell} z(x) e^{-i2\pi kx} dx \right|^2 \quad . \tag{B.5}$$

The inverse Fourier transform of $\Phi_{\ell z}(k)$ was computed next, which yielded the sample autocorrelation function of the "clipped" waveform:

$$R_{\ell z}(\xi) = \int \Phi_{\ell z}(k) e^{i2\pi kx} dk \quad . \tag{B.6}$$

The sample autocorrelation function corrected for finite sample length is, therefore, $R_{\ell z}(\xi)/(1 - \frac{|\xi|}{2\ell-M})$. The autocorrelation function correction of Eq. (3.20) was then used to correct the autocorrelation function of the clipped signal; i.e.,

$$R_{\ell}'(\xi) = \sin\left[\frac{\pi}{2}\,\frac{R_{\ell z}(\xi)}{(1 - \frac{|\xi|}{2\ell-M})}\right] \tag{B.7}$$

was formed. The Fourier transform $p_0(\xi)$, given by Eq. (B.1) of the frequency smoothing function, was then multiplied by $R_{\ell}'(\xi)$, and the Fourier transform of the resulting product was then taken:

$$\Phi_p(k) = \sigma^2 \int_{-M}^{M} p_0(\xi)\; R_{\ell}'(\xi) e^{-i2\pi k\xi} d\xi \tag{B.8}$$

The function $\Phi_p(k)$ is the spectrum obtained from the infinitely clipped signal, and corrected for the clipping operation. The values of $\Phi_p(k)$, computed in the above manner, are shown in Fig. 4.

APPENDIX C
COMPUTATIONAL PROCEDURE FOR SMOOTHING OF von KARMAN TRANSVERSE SPECTRUM

The unnormalized form of the (two-sided) von Karman transverse spectrum is

$$\Phi_K(k) = \sigma^2 L \frac{1 + 188.75\ L^2k^2}{[1 + 70.78\ L^2k^2]^{11/6}} \quad , \tag{C.1}$$

$$\int_{-\infty}^{\infty} \Phi(k)dk = \sigma^2 \quad , \tag{C.2}$$

where k is wavenumber in cycles/unit distance, and L is the integral scale. The exact values of the constants in Eq. (C.1) are the left-hand sides of

$$25\pi \left[\frac{\Gamma(4/3)}{\Gamma(11/6)}\right]^2 = 70.78 \quad , \tag{C.3}$$

and

$$\frac{200}{3}\pi \left[\frac{\Gamma(4/3)}{\Gamma(11/6)}\right]^2 = 188.75 \quad , \tag{C.4}$$

where $\Gamma(\cdot)$ is the gamma function.

We wish to compute a wavenumber smoothed version of the von Karman transverse spectrum, where the wavenumber smoothing function must be that used in smoothing the empirically determined spectra, as described in Appendix B. To accomplish this, we shall use the fact that the inverse Fourier transform of the smoothing operation (a convolution) is the product of the (inverse) transform of the smoothing function and the (inverse) transform of the von Karman spectrum — this latter quantity being the autocorrelation function,

$$\phi_K(\xi) = \int_{-\infty}^{\infty} \Phi_K(k)e^{i2\pi k\xi}dk \quad . \tag{C.5}$$

Thus, using $p_0(\xi)$ as defined by Eq. (B.1), we computed the smoothed von Karman spectrum $\Phi_S(k)$ using

$$\Phi_s(k) = \int_{-M}^{M} p_0(\xi)\ \phi_K(\xi) e^{-i2\pi k\xi} d\xi \quad . \tag{C.6}$$

This smoothed spectrum is a function of the variance σ^2 and integral scale L of the turbulence. However, using the method outlined below, it is necessary to calculate the inverse Fourier transform of the von Karman spectrum only once.

Define a dimensionless wavenumber by

$$\bar{k} \triangleq Lk \quad , \tag{C.7}$$

and define a normalized von Karman transverse spectrum by

$$\bar{\Phi}(\bar{k}) \triangleq \frac{1 + 188.75\ \bar{k}^2}{[1 + 70.78\ \bar{k}^2]^{11/6}} \quad . \tag{C.8}$$

According to Eqs. (C.1), (C.7), and (C.8), the unnormalized von Karman spectrum may be expressed in terms of the normalized spectrum by

$$\Phi_K(k) = \sigma^2 L\ \bar{\Phi}(Lk) \quad . \tag{C.9}$$

Define, as in Eq. (C.5), the von Karman transverse autocorrelation function by

$$\begin{aligned} \phi_K(\xi) &= \int_{-\infty}^{\infty} \Phi_K(k) e^{i2\pi k\xi} dk \\ &= \sigma^2 L \int_{-\infty}^{\infty} \bar{\Phi}(Lk) e^{i2\pi Lk\xi/L} dk \\ &= \sigma^2 \int_{-\infty}^{\infty} \bar{\Phi}(\bar{k}) e^{i2\pi\bar{k}\xi/L} d\bar{k} \quad . \end{aligned} \tag{C.10}$$

If we define a normalized length measure by

$$\bar{\xi} \triangleq \xi/L \quad , \tag{C.11}$$

and a normalized von Karman transverse autocorrelation function by

$$\overline{\phi}(\overline{\xi}) \triangleq \int_{-\infty}^{\infty} \overline{\Phi}(\overline{k}) e^{i2\pi\overline{k}\overline{\xi}} d\overline{k} \quad , \tag{C.12}$$

then, according to Eqs. (C.10) to (C.12), we may express $\phi_K(\xi)$ as

$$\phi_K(\xi) = \sigma^2 \, \overline{\phi}(\xi/L) \quad . \tag{C.13}$$

From Eq. (C.11), we have

$$\xi = L\overline{\xi} \quad . \tag{C.14}$$

It follows that computation of $\overline{\phi}(\overline{\xi})$, using Eq. (C.12), at values of $\overline{\xi} = n\Delta\overline{\xi}$, n = 0,1,2,··· provides values of $\phi_K(\xi)$ at values of $n\Delta\xi = nL\Delta\overline{\xi}$, n = 0,1,2,···, i.e.,

$$\phi_K(nL\Delta\overline{\xi}) = \sigma^2 \, \overline{\phi}(n\Delta\overline{\xi}) \quad , \qquad n = \text{any integer} \quad . \tag{C.15}$$

The relationship of Eq. (C.15) was used to compute $\phi_K(\xi)$ for different values of the integral scale L.

Families of smoothed von Karman transverse spectra are displayed in Fig. C.1, C.2, and C.3 for values of M = 609.6 m (2000 ft), 1219.2 m (4000 ft), and infinity (which corresponds to no smoothing).

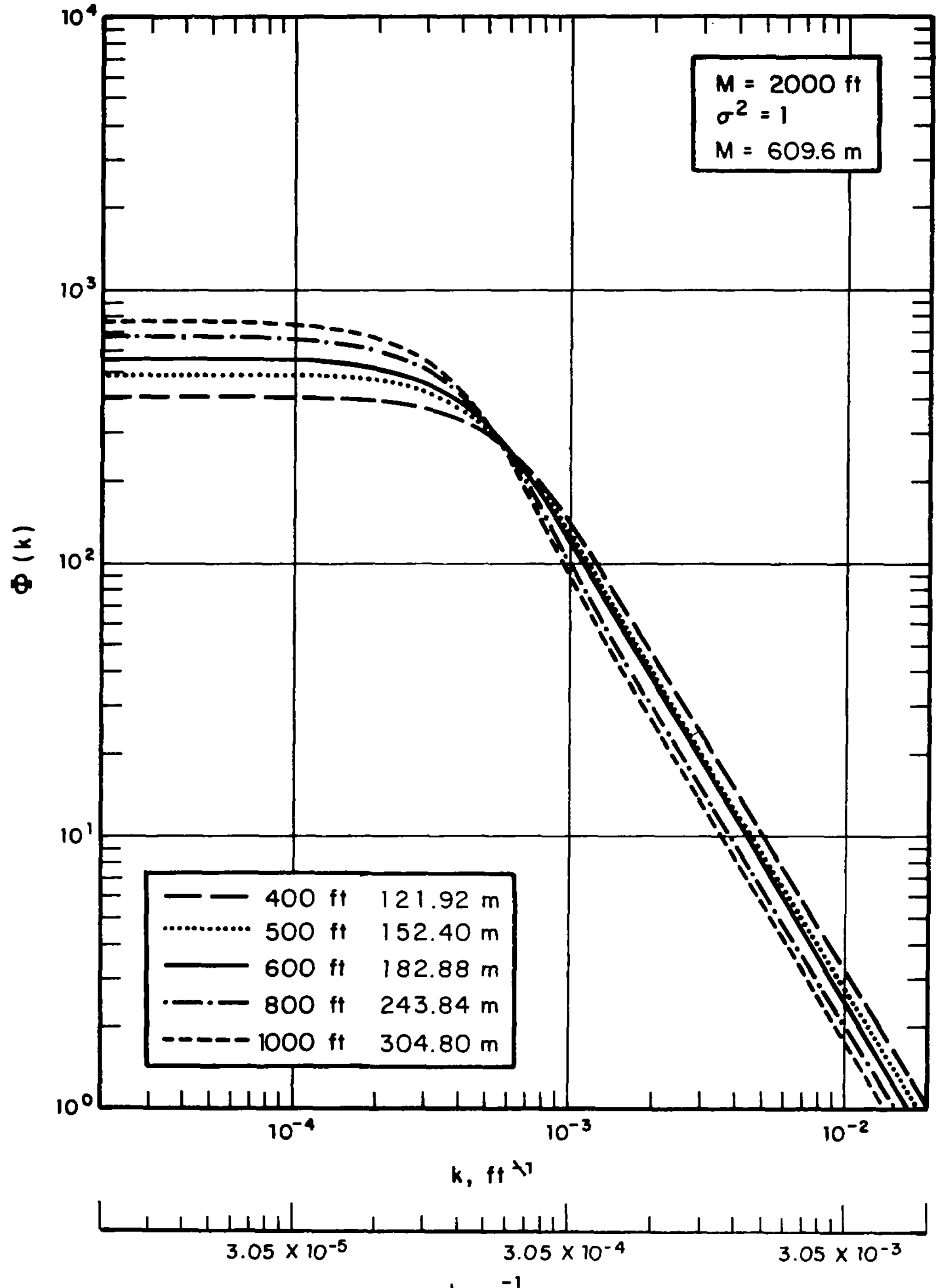

FIG. C.1. FAMILY OF von KARMAN TRANSVERSE SPECTRA SMOOTHED WITH PAPOULIS WINDOW [M = 609.6 m (2000 ft)].

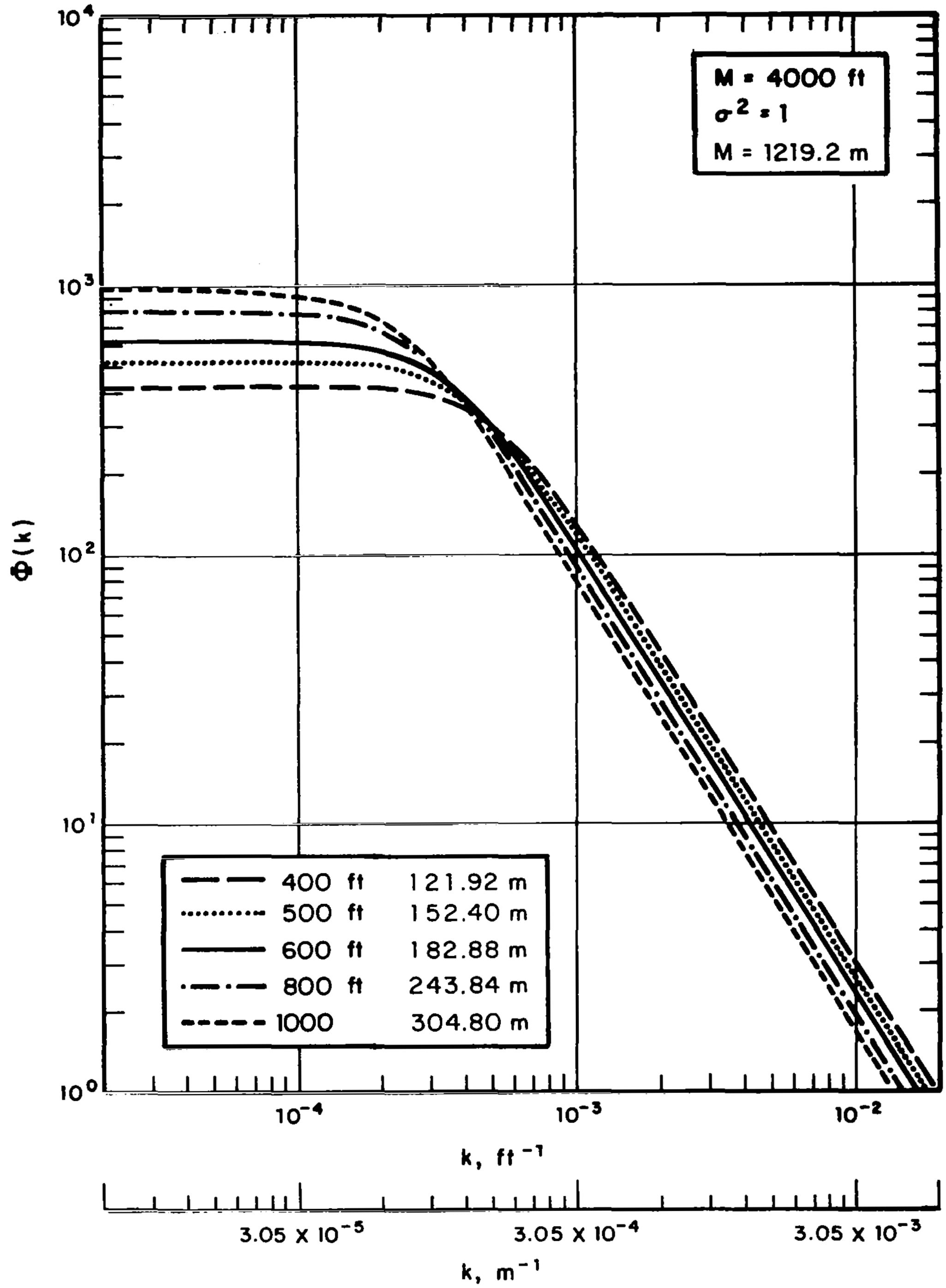

FIG. C.2. FAMILY OF von KARMAN TRANSVERSE SPECTRA SMOOTHED WITH PAPOULIS WINDOW [M = 1219.2 m (4000 ft)].

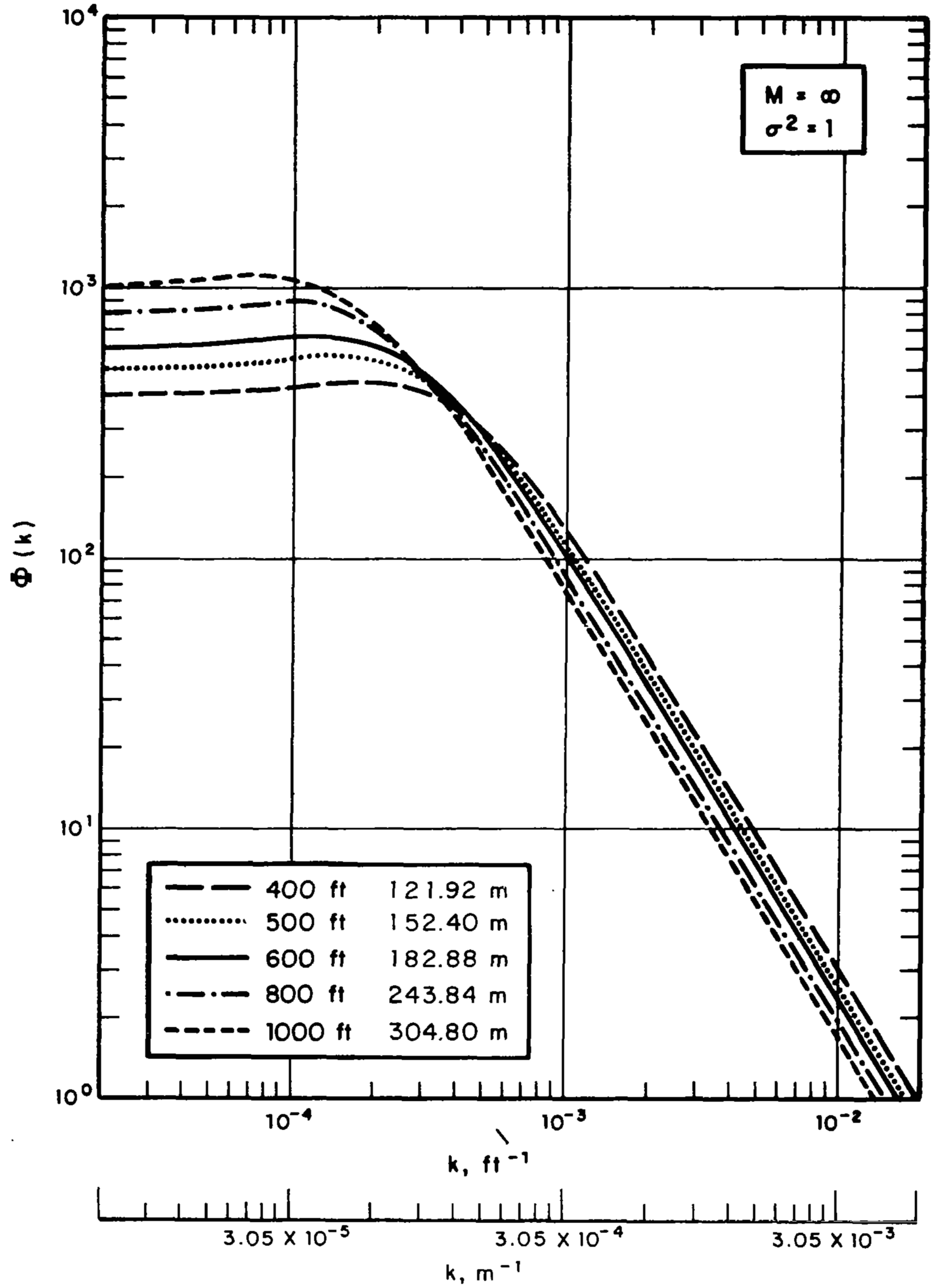

FIG. C.3. FAMILY OF UNSMOOTHED von KARMAN TRANSVERSE SPECTRA (M = INFINITY).

APPENDIX D
STATISTICAL CONFIDENCE OF VARIANCE ESTIMATES

The stochastic process w(x) may be regarded as the result of passing white Gaussian noise through a filter whose squared frequency-response-function magnitude has the form of the power spectrum of the turbulence velocities; i.e., the von Karman transverse spectrum of Eq. (C.1). Consequently, we may use the well known relationship; e.g., Bendat and Piersol [6], p. 260 to 265.

$$\frac{\sigma}{m} = \frac{1}{\sqrt{BT}} \quad , \tag{D.1}$$

to compute the relative standard deviation of an estimate of the mean-square value of the process w(x); i.e., an estimate of the quantity $\sigma^2(x)$ of Eq. (3.1). Equation (D.1), applied to the present situation, should be written as

$$\frac{\sigma}{m} = \frac{1}{\sqrt{\Delta k \Delta x}} \quad , \tag{D.2}$$

where Δk is the "effective bandwidth" of the wavenumber spectrum of the turbulence, measured in cycles per unit length, and Δx is the averaging interval as used in Eqs. (3.24) and (3.25). The proper definition of Δk for use in Eq. (D.2) is, see Bendat and Piersol [6], p. 265.

$$\Delta k = \frac{\left[\int_0^\infty \Phi(k)dk\right]^2}{\int_0^\infty \Phi^2(k)dk} \quad , \tag{D.3}$$

where, in the present application, $\Phi(k)$ is the von Karman transverse spectrum, given by Eq. (C.1).

Transverse velocity components. From Eqs. (D.3) and (C.2), it is evident that Δk may be expressed as

$$\Delta k = \frac{\left(\frac{\sigma^2}{2}\right)^2}{\int_0^\infty \Phi_K^2(k)dk}$$

$$= \frac{1}{\frac{4}{\sigma^4}\int_0^\infty \Phi_K^2(k)dk} \quad ; \tag{D.4}$$

hence, from Eq. (C.1), we have

$$(\Delta k)^{-1} = 4L^2 \int_0^\infty \frac{[1 + 188.75\ L^2k^2]^2}{[1 + 70.78\ L^2k^2]^{2\,2/6}}\,dk$$

$$= \frac{4L}{\sqrt{70.78}} \int_0^\infty \frac{(1 + \frac{8}{3}\,x^2)^2}{(1 + x^2)^{2\,2/6}}\,dx$$

$$= \frac{4L\ \Gamma(\frac{11}{6})}{5\sqrt{\pi}\ \Gamma(\frac{4}{3})} \int_0^\infty \frac{(1 + \frac{8}{3}\,x^2)^2}{(1 + x^2)^{2\,2/6}}\,dx \quad , \tag{D.5}$$

where we have substituted

$$x = \sqrt{70.78}\ Lk$$

$$= 5\sqrt{\pi}\,\frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{11}{6})}\,Lk \quad , \tag{D.6}$$

according to Eqs. (C.3) and (C.4). Denoting the integral in the right-hand side of Eq. (D.5) by I, we have

$$I = \int_0^\infty \frac{(1 + \frac{16}{3}\,x^2 + \frac{64}{9}\,x^4)}{(1 + x^2)^{2\,2/6}}\,dx \quad . \tag{D.7}$$

Using formula 3.251.2 on p. 295 of Gradshteyn and Ryzhik [18], we have for I:

$$I = \frac{1}{2} B(\tfrac{1}{2}, \tfrac{19}{6}) + \frac{16}{6} B(\tfrac{3}{2}, \tfrac{13}{6}) + \frac{32}{9} B(\tfrac{5}{2}, \tfrac{7}{6})$$

$$= \frac{1}{2} \frac{\Gamma(\frac{1}{2})\ \Gamma(\frac{19}{6})}{\Gamma(\frac{22}{6})} + \frac{16}{6} \frac{\Gamma(\frac{3}{2})\ \Gamma(\frac{13}{6})}{\Gamma(\frac{22}{6})} + \frac{32}{9} \frac{\Gamma(\frac{5}{2})\ \Gamma(\frac{7}{6})}{\Gamma(\frac{22}{6})} \quad , \tag{D.8}$$

where

$$B(x,y) = \frac{\Gamma(x)\ \Gamma(y)}{\Gamma(x+y)} \tag{D.9}$$

is the beta function, and $\Gamma(\cdot)$ is the gamma function. Using

$$\Gamma(n+1) = n\ \Gamma(n) \quad , \tag{D.10}$$

and

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad , \tag{D.11}$$

we have

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2} \sqrt{\pi} \tag{D.12}$$

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\ \Gamma(\tfrac{3}{2})$$

$$= \tfrac{3}{4} \sqrt{\pi} \quad , \tag{D.13}$$

and

$$\Gamma(\tfrac{13}{6}) = \tfrac{7}{6}\ \Gamma(\tfrac{7}{6}) \tag{D.14}$$

$$\Gamma(\tfrac{19}{6}) = \tfrac{13}{6}\ \Gamma(\tfrac{13}{6})$$

$$= \tfrac{91}{36}\ \Gamma(\tfrac{7}{6}) \quad , \tag{D.15}$$

and

$$\Gamma(\tfrac{22}{6}) = \Gamma(\tfrac{11}{3})$$

$$= \frac{8}{3}\,\Gamma(\tfrac{8}{3})$$

$$= \frac{40}{9}\,\Gamma(\tfrac{5}{3}) \quad . \tag{D.16}$$

Substituting Eqs. (D.11) to (D.16) into (D.8) and simplifying the result gives

$$I = \sqrt{\pi}\,\frac{79\;\Gamma(\frac{7}{6})}{64\;\Gamma(\frac{5}{3})} \quad . \tag{D.17}$$

Finally, combining Eqs. (D.5) and (D.17) yields

$$\Delta k^{-1} = L\,\frac{79\;\Gamma(\frac{7}{6})\;\Gamma(\frac{11}{6})}{80\;\Gamma(\frac{4}{3})\;\Gamma(\frac{5}{3})} \quad . \tag{D.18}$$

From Dwight [*19*], pp. 132-133, we find

$$\Gamma(\tfrac{7}{6}) = 0.92772$$

$$\Gamma(\tfrac{11}{6}) = 0.94066$$

$$\Gamma(\tfrac{4}{3}) = 0.89298$$

$$\Gamma(\tfrac{5}{3}) = 0.90275 \quad . \tag{D.19}$$

Combining Eqs. (D.18) and (D.19) yields, finally,

$$\Delta k^{-1} = 1.0690\;L \quad . \tag{D.20}$$

Combining Eqs. (D.2) and (D.20) gives, for the relative variance of an estimate of the mean-square value of the transverse velocity component,

$$\left(\frac{\sigma}{m}\right)^2 = 1.0690\ L/(\Delta x) \quad . \tag{D.21}$$

Longitudinal velocity component. The von Karman (two-sided) spectrum of the longitudinal velocity component may be expressed as

$$\Phi(k) = 2\sigma_u^2 L \frac{1}{[1 + 70.78\ L^2k^2]^{5/6}} \quad . \tag{D.22}$$

Using Eq. (D.3) and the fact that

$$\sigma_u^2 = 2 \int_0^\infty \Phi(k)dk \quad , \tag{D.23}$$

we have, for the present case,

$$\begin{aligned}(\Delta k)^{-1} &= 16L^2 \int_0^\infty \frac{1}{[1 + 70.78\ L^2k^2]^{5/3}}\, dk \\ &= \frac{16L}{\sqrt{70.78}} \int_0^\infty \frac{1}{(1 + x^2)^{5/3}}\, dx \\ &= \frac{16L\ \Gamma(\frac{11}{6})}{5\sqrt{\pi}\ \Gamma(\frac{4}{3})} \int_0^\infty \frac{1}{(1 + x^2)^{5/3}}\, dx \quad ,\end{aligned} \tag{D.24}$$

where, again, we have used Eq. (D.6). The above integral may be evaluated by formula 3.251.2 on p. 295 of Gradshteyn and Ryzhik [*18*]:

$$\begin{aligned}\int_0^\infty \frac{1}{(1 + x^2)^{5/3}}\, dx &= \frac{1}{2}\, B(\tfrac{1}{2}, \tfrac{7}{6}) \\ &= \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})} \quad .\end{aligned} \tag{D.25}$$

Combining Eqs. (D.24) and (D.25), we have

$$(\Delta k)^{-1} = \frac{8L\ \Gamma(\frac{7}{6})\ \Gamma(\frac{11}{6})}{5\ \Gamma(\frac{4}{3})\ \Gamma(\frac{5}{3})} = \sqrt{3}\ L \tag{D.26}$$

$$= 1.7320\ L \quad . \tag{D.27}$$

Consequently, the relative variance of an estimate of the mean-square value of the longitudinal velocity component is

$$(\frac{\sigma}{m})^2 = 1.7320\ L/(\Delta x) \quad . \tag{D.28}$$

APPENDIX E

EXPANSION COEFFICIENTS FOR DETERMINISTIC MODULATING FUNCTIONS

Expressions* for $a_0(x)$ to $a_8(x)$, obtained from Eq. (4.22), are written out below in terms of the derivatives

$$\sigma^{(n)}(x) \triangleq \frac{d^n \sigma(x)}{dx^n} \quad .$$

$$a_0(x) = [\sigma(x)]^2$$

$$a_2(x) = -\frac{1}{8\pi^2}\{\sigma(x)\,\sigma^{(2)}(x) - [\sigma^{(1)}(x)]^2\}$$

$$a_4(x) = \frac{1}{128\pi^4}\{\sigma(x)\,\sigma^{(4)}(x) - 4\sigma^{(1)}(x)\,\sigma^{(3)}(x) + 3\,[\sigma^{(2)}(x)]^2\}$$

$$a_6(x) = -\frac{1}{2048\pi^6}\{\sigma(x)\,\sigma^{(6)}(x) - 6\sigma^{(1)}(x)\,\sigma^{(5)}(x)$$

$$+ 15\sigma^{(2)}(x)\,\sigma^{(4)}(x) - 10\,[\sigma^{(3)}(x)]^2\}$$

$$a_8(x) = \frac{1}{32768\pi^8}\{\sigma(x)\,\sigma^{(8)}(x) - 8\sigma^{(1)}(x)\,\sigma^{(7)}(x)$$

$$+ 28\sigma^{(2)}(x)\,\sigma^{(6)}(x) - 56\sigma^{(3)}(x)\,\sigma^{(5)}(x)$$

$$+ 35\,[\sigma^{(4)}(x)]^2\} \quad .$$

*For n = odd integer, all $a_n(x)$ are zero.

APPENDIX F

EXPANSION COEFFICIENTS FOR ERGODIC MODULATING PROCESSES

For ergodic modulating process $\sigma(x)$, we need to evaluate integrals of the expansion coefficients $a_n(x)$, where $a_n(x)$ is given by Eq. (4.22) for n = even, and $a_n(x) = 0$ for n = odd.

Denote the *kth* derivative of $\sigma(x)$ by $\sigma^{(k)}(x)$, i.e.,

$$\sigma^{(k)}(x) \triangleq \frac{d^k \sigma(x)}{dx^k} \quad . \tag{F.1}$$

Integrating by parts a typical term in Eq. (4.22) gives

$$\int_A^B \sigma^{(k)}(x)\, \sigma^{(n-k)}(x)dx =$$

$$= \sigma^{(k)}(x)\, \sigma^{(n-k-1)}(x)\Big|_{x=A}^{x=B} - \int_A^B \sigma^{(k+1)}(x)\, \sigma^{(n-k-1)}(x)dx \quad . \tag{F.2}$$

Letting k' = k + 1, we see that the integral in the right-hand side of Eq. (F.2) may be expressed as

$$\int_A^B \sigma^{(k+1)}(x)\, \sigma^{(n-k-1)}(x)dx = \int_A^B \sigma^{(k')}(x)\, \sigma^{(n-k')}(x)dx \quad , \tag{F.3}$$

which has the same form as the left-hand side of Eq. (F.2) (with k = k'). Hence, we have

$$\int_A^B \sigma^{(k)}(x)\ \sigma^{(n-k)}(x)dx =$$

$$= \sigma^{(k)}(x)\ \sigma^{(n-k-1)}(x)\Big|_{x=A}^{x=B} - \int_A^B \sigma^{(k')}(x)\ \sigma^{(n-k')}(x)dx$$

$$= \sigma^{(k)}(x)\ \sigma^{(n-k-1)}(x)\Big|_{x=A}^{x=B} - \sigma^{(k+1)}(x)\ \sigma^{(n-k-2)}(x)\Big|_{x=A}^{x=B} +$$

$$+ \int_A^B \sigma^{(k'')}(x)\ \sigma^{(n-k'')}(x)dx \quad , \tag{F.4}$$

where $k'' = k' + 1 = k + 2$. We may continue to integrate by parts in this fashion, each time obtaining an integral of the form $\int_A^B \sigma^{(k+m)}(x)\ \sigma^{(n-k-m)}(x)dx$ on the right-hand side. We wish to terminate this procedure at the value of m where

$$k + m = n - k - m \quad , \tag{F.5}$$

i.e., where

$$m = \frac{n}{2} - k \quad . \tag{F.6}$$

For this value of m, we have

$$k + m = n - k - m = \frac{n}{2} \quad , \tag{F.7}$$

which is an integer since n = even. Consequently, repeated integration by parts of Eq. (F.4) gives

$$\int_A^B \sigma^{(k)}(x)\ \sigma^{(n-k)}(x)dx =$$

$$= \sum_{j=0}^{\frac{n}{2}-k-1} (-1)^j\ \sigma^{(k+j)}(x)\ \sigma^{(n-k-j-1)}(x)\Big|_{x=A}^{x=B} +$$

$$+ (-1)^{\frac{n}{2}-k} \int_A^B [\sigma^{(n/2)}(x)]^2 dx \quad . \tag{F.8}$$

Integrating Eq. (4.22) over the interval A < x < B, we therefore have

$$\int_A^B a_n(x)dx = \frac{1}{(i4\pi)^n}$$

$$\times \Bigg\{ 2\left[\sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n}{k} \sum_{j=0}^{\frac{n}{2}-k-1} (-1)^j\ \sigma^{(k+j)}(x)\ \sigma^{(n-k-j-1)}(x)\Big|_{x=A}^{x=B}\right]$$

$$+ 2\left[\sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n}{k} (-1)^{\frac{n}{2}-k} \int_A^B [\sigma^{(n/2)}(x)]^2 dx\right]$$

$$+ (-1)^{n/2} \binom{n}{n/2} \int_A^B [\sigma^{(n/2)}(x)]^2 dx \Bigg\} \tag{F.9}$$

Consider the double summation in the above expression first. Letting $j = \ell - k$, and hence, $\ell = j + k$ in the inner sum, we have

$$\sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n}{k} \sum_{j=0}^{\frac{n}{2}-k-1} (-1)^j \; \sigma^{(k+j)}(x) \; \sigma^{(n-k-j-1)}(x)$$

$$= \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} \sum_{j=0}^{\frac{n}{2}-1-k} (-1)^{j+k} \; \sigma^{(j+k)}(x) \; \sigma^{[n-1-(j+k)]}(x)$$

$$= \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} \sum_{\ell=k}^{\frac{n}{2}-1} (-1) \; \sigma^{(\ell)}(x) \; \sigma^{(n-1-\ell)}(x) \tag{F.10}$$

However, it is immediately evident from Fig. F.1 that for an arbitrary function $G(k,\ell)$ defined for integer arguments k and ℓ, we have

$$\sum_{k=0}^{\frac{n}{2}-1} \sum_{\ell=k}^{\frac{n}{2}-1} G(k,\ell) = \sum_{\ell=0}^{\frac{n}{2}-1} \sum_{k=0}^{\ell} G(k,\ell) \quad .$$

Hence, the right hand side of Eq. (F.10) may be further simplified. Defining

$$F(\ell;n) \triangleq \sum_{k=0}^{\ell} \binom{n}{k} \quad , \tag{F.11}$$

we have for the desired expression

$$\sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} \sum_{\ell=k}^{\frac{n}{2}-1} (-1)^\ell \; \sigma^{(\ell)}(x) \; \sigma^{(n-1-\ell)}(x) =$$

$$\sum_{\ell=0}^{\frac{n}{2}-1} (-1)^\ell \; F(\ell;n) \; \sigma^{(\ell)}(x) \; \sigma^{(n-1-\ell)}(x) \quad . \tag{F.12}$$

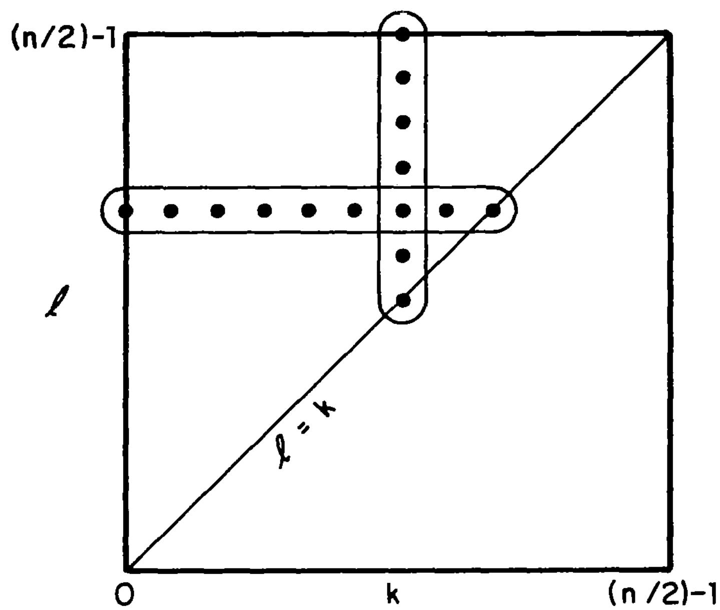

FIG. F.1. ILLUSTRATING INTERCHANGE OF ORDER OF SUMMATION.

Furthermore, considering the two terms involving integrals over x in the right-hand side of Eq. (F.9), we have

$$2\left[\sum_{k=0}^{\frac{n}{2}-1}(-1)^k \binom{n}{k} (-1)^{\frac{n}{2}-k} \int_A^B [\sigma^{(n/2)}(x)]^2 dx\right]$$

$$+ (-1)^{n/2} \binom{n}{n/2} \int_A^B [\sigma^{(n/2)}(x)]^2 dx$$

$$= (-1)^{n/2} \left\{2\left[\sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k}\right] + \binom{n}{n/2}\right\} \int_A^B [\sigma^{(n/2)}(x)]^2 dx$$

$$= (-1)^{n/2} \left[\sum_{k=0}^{n} \binom{n}{k}\right] \int_A^B [\sigma^{(n/2)}(x)]^2 dx$$

$$= (2i)^n \int_A^B [\sigma^{(n/2)}(x)]^2 dx \quad , \tag{F.13}$$

where we have used the fact that n is even and also Eq. (4.21) in going to the next to the last step, and where, in going to the last step, we have used $(-1)^{\frac{1}{2}} = i$ and

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad ,$$

which follows from the binomial expansion,

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k} \quad ,$$

by setting u = 1 and v = 1. Thus, combining Eqs. (F.9, F.10, F.12, and F.13), we have, finally,

$$\int_A^B a_n(x)dx = \frac{2}{(i4\pi)^n}\left[\sum_{\ell=0}^{\frac{n}{2}-1} (-1)^{\ell}\, F(\ell;n)\, \sigma^{(\ell)}(x)\, \sigma^{(n-1-\ell)}(x)\right]\Bigg|_{x=A}^{x=B}$$

$$+ \frac{1}{(2\pi)^n}\int_A^B [\sigma^{(n/2)}(x)]^2 dx \quad , \tag{F.14}$$

where n is even, and where we remind the reader of the definitions of $\sigma^{(k)}(x)$ and $F(\ell;n)$ given by Eqs. (F.1) and (F.11) respectively.

Equation (F.14) is an exact expression for the integral of the expansion coefficients $a_n(x)$. It consists of two terms. The first term involves derivatives of $\sigma(x)$ from orders zero to $n-1$, evaluated at the endpoints of the interval of integration. The second term involves an integral of the square of the $(n/2)$*th* derivative of $\sigma(x)$.

APPENDIX G

DERIVATIVES REQUIRED FOR EVALUATION OF EXPANSION FUNCTIONS

After considerable algebraic manipulation, the first eight derivatives of the function $f_2(\bar{k})$, required for evaluation of Eq. (4.63), may be obtained as

$$f_2(\bar{k}) = (1 + C\,\bar{k}^2)^{-1\,1/6}$$

$$f_2^{(1)}(\bar{k}) = -\frac{11}{3} C\,\bar{k}(1 + C\,\bar{k}^2)^{-1\,7/6}$$

$$f_2^{(2)}(\bar{k}) = -\frac{11}{3} C[(1 + C\,\bar{k}^2)^{-1\,7/6} - \frac{17}{3} C\,\bar{k}^2(1 + C\,\bar{k}^2)^{-2\,3/6}]$$

$$f_2^{(3)}(\bar{k}) = \frac{187}{3} C^2\,\bar{k}[(1 + C\,\bar{k}^2)^{-2\,3/6} - \frac{23}{9} C\,\bar{k}^2(1 + C\,\bar{k}^2)^{-2\,9/6}]$$

$$f_2^{(4)}(\bar{k}) = \frac{187}{3} C^2[(1 + C\,\bar{k}^2)^{-2\,3/6} - \frac{46}{3} C\,\bar{k}^2(1 + C\,\bar{k}^2)^{-2\,9/6} + \frac{667}{27} C^2\,\bar{k}^4(1 + C\,\bar{k}^2)^{-3\,5/6}]$$

$$f_2^{(5)}(\bar{k}) = -\frac{21505}{9} C^3\,\bar{k}[(1 + C\,\bar{k}^2)^{-2\,9/6} - \frac{58}{9} C\,\bar{k}^2(1 + C\,\bar{k}^2)^{-3\,5/6} + \frac{203}{27} C^2\,\bar{k}^4(1 + C\,\bar{k}^2)^{-4\,1/6}]$$

$$f_2^{(6)}(\bar{k}) = -\frac{21505}{9} C^3[(1 + C\,\bar{k}^2)^{-2\,9/6} - 29\,C\,\bar{k}^2(1 + C\,\bar{k}^2)^{-3\,5/6} + \frac{1015}{9} C^2\,\bar{k}^4(1 + C\,\bar{k}^2)^{-4\,1/6} - \frac{8323}{81} C^3\,\bar{k}^6(1 + C\,\bar{k}^2)^{-4\,7/6}]$$

$$f_2^{(7)}(\bar{k}) = \frac{4365515}{27} C^4\,\bar{k}[(1 + C\,\bar{k}^2)^{-3\,5/6} - \frac{35}{3} C\,\bar{k}^2(1 + C\,\bar{k}^2)^{-4\,1/6} + \frac{287}{9} C^2\,\bar{k}^4(1 + C\,\bar{k}^2)^{-4\,7/6} - \frac{1927}{81} C^3\,\bar{k}^6(1 + C\,\bar{k}^2)^{-5\,3/6}]$$

$$f_2^{(8)}(\bar{k}) = \frac{4365515}{27} C^4[(1 + C\,\bar{k}^2)^{-3\,5/6} - \frac{140}{3} C\,\bar{k}^2(1 + C\,\bar{k}^2)^{-4\,1/6} + \frac{2870}{9} C^2\,\bar{k}^4(1 + C\,\bar{k}^2)^{-4\,7/6} - \frac{53956}{81} C^3\,\bar{k}^6(1 + C\,\bar{k}^2)^{-5\,3/6} + \frac{102131}{243} C^4\,\bar{k}^8(1 + C\,\bar{k}^2)^{-5\,9/6}] \quad .$$

APPENDIX H
SPECTRUM COMPUTATIONS FOR MODEL OF ABRUPT ONSET OF TURBULENCE

Here, we derive an expression for $a_2(x)$, given by Eq. (4.52a), from the expressions for $\sigma(x)$, $\sigma'(x)$, and $\sigma''(x)$ given by Eqs. (4.66) and (4.67). Define

$$y = \frac{2x}{L_\sigma} \quad . \tag{H.1}$$

From Eqs. (4.66) and (4.67b), we have

$$\sigma(x)\ \sigma''(x) = -\frac{2}{L_\sigma^2} \operatorname{sech}^2 y(\tanh y + \tanh^2 y) \quad ; \tag{H.2}$$

whereas, from Eq. (4.67a), we have

$$[\sigma'(x)]^2 = \frac{1}{L_\sigma^2} \operatorname{sech}^4 y \quad ; \tag{H.3}$$

hence,

$$\sigma(x)\ \sigma''(x) - [\sigma'(x)]^2 = -\frac{2}{L_\sigma^2} \operatorname{sech}^2 y(\tanh y + \tanh^2 y + \frac{1}{2} \operatorname{sech}^2 y) \tag{H.4}$$

However,

$$\operatorname{sech}^2 y = 1 - \tanh^2 y \quad ; \tag{H.5}$$

therefore,

$$\begin{aligned} \tanh y + \tanh^2 y + \frac{1}{2} \operatorname{sech}^2 y &= \tanh y + \tanh^2 y + \frac{1}{2} - \frac{1}{2} \tanh^2 y \\ &= \frac{1}{2} (\tanh^2 y + 2 \tanh y + 1) \\ &= \frac{1}{2} (1 + \tanh y)^2 \\ &= 2\ \sigma^2(x) \quad , \end{aligned} \tag{H.6}$$

where, in going to the last step, we have used Eq. (4.66). Combining Eqs. (H.4) and (H.6) with Eq. (4.52a) gives

$$a_2(x) = \frac{\sigma^2(x)}{2\pi^2 \; L_\sigma^2} \operatorname{sech}^2 \, (2x/L_\sigma) \quad , \tag{H.7}$$

which is the result entered as Eq. (4.68).

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